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A NEAR SUBTRACTION RESULT IN METRIC SPACES (THE LOCALLY CLOSED GRAPH CASE)

CORNELIU URSESCU

Abstract. The paper concerns a near subtraction result for multifunctions with locally closed graphs in metric spaces.

1. Introduction

Let X and Y be topological spaces, let $F : X \to Y$ be a multifunction, and recall the multifunction F is said to be *open* if the set $F(U)$ is open in Y whenever the set U is open in X. This property can be analyzed through a pointwise property. The multifunction F is said to be *open at a point* $(x, y) \in \text{graph}(F)$ if the set $F(U)$ is a neighborhood of y whenever the set U is a neighborhood of x. Clearly, F is open iff F is open at each point $(x, y) \in \text{graph}(F)$. A weaker version of pointwise openness can be defined by using set closure. The multifunction F is said to be *nearly open at a point* $(x, y) \in \text{graph}(F)$ if the set $\overline{F(U)}$ is a neighborhood of y whenever the set U is a neighborhood of x. Here, $\overline{S}$ stands for the *closure* of the set S. Pointwise near openness can be used to synthesize near openness, a weaker version of openness. The multifunction F said to be *nearly open* if F is nearly open at each point $(x, y) \in \text{graph}(F)$.

In some cases, openness at a point can be derived from near openness at that point (see [13, p. 439, Lemma 3] and [20, Theorems 1 and 2], where the multifunction has a locally closed graph).

In other cases, such pointwise results may fail even if the multifunction has a closed graph (see the first counterexample in Section 6), but some special properties of openness can be derived from their near versions.

In most of these cases, the special property of openness and its near version imply at least local uniform openness and local uniform near openness respectively. Let us describe these local notions, which make sense in uniform spaces.

First, we consider setwise openness and setwise near openness. The multifunction F is said to be *open on a set* $S \subseteq \text{graph}(F)$ if F is open

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at every point $(x, y) \in S$, whereas the multifunction F is said to be *nearly open on a set* $S \subseteq \mathrm{graph}(F)$ if F is nearly open at every point $(x, y) \in S$.

Further, we note openness and near openness of F at a point (x, y) can be rephrased by using any base $\mathcal{U}$ for the neighborhood system of x as well as any base $\mathcal{V}$ for the neighborhood system of y. Namely, F is open at (x, y) iff for every $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that

$$V \subseteq F(U),$$

whereas F is nearly open at (x, y) iff for every $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that

$$V \subseteq \overline{F(U)}.$$

Now, let X and Y be uniform spaces, and note F is open at (x, y) iff for every entourage U on X there exists an entourage V on Y such that

$$V[y] \subseteq F(U[x]), \tag{1}$$

whereas F is nearly open at (x, y) iff for every entourage U on X there exists an entourage V of y such that

$$V[y] \subseteq \overline{F(U[x])}. \tag{2}$$

Here, $E[p]$ stands for the set of all points p' such that (p, p') belongs to the entourage E.

Further, we consider the uniform versions of setwise openness and of setwise near openness. The multifunction F is said to be *uniformly open on a set* $S \subseteq \mathrm{graph}(F)$ if for every entourage U on X there exists an entourage V on Y such that for every $(x, y) \in S$ there holds inclusion (1), whereas the multifunction F is said to be *uniformly nearly open* on a set $S \subseteq \mathrm{graph}(F)$ if for every entourage U on X there exists an entourage V on Y such that for every $(x.y) \in S$ there holds inclusion (2).

Clearly, uniform openness on a set implies uniform near openness on that set. In some general spaces, uniform openness of F on $\mathrm{graph}(F)$ can be derived from uniform near openness of F on $\mathrm{graph}(F)$ (see [9, p. 202, Lemma 36], where F has a closed graph and the uniformity on X has a countable base, i.e. the uniform space X is metrizable; see also [9, p. 214, Note], where a counterexample shows the result may fail if X is not metrizable). In some particular spaces, uniform openness of F on a set $S \subseteq \mathrm{graph}(F)$ can be derived from uniform near openness of F on S (see [18, p. 146, Theorem 6], where X and Y are metric spaces).

Finally, we consider the local notions of uniform openness and of uniform near openness. The multifunction F is said to be *locally uniformly open* if there exists an open covering $\mathcal{W}$ of $\mathrm{graph}(F)$ such that F is uniformly open on $W \cap \mathrm{graph}(F)$ whenever $W \in \mathcal{W}$, whereas the multifunction F is said to be *locally uniformly nearly open* if there exists an open covering $\mathcal{W}$ of graph(F) such that F is uniformly nearly open on $W \cap \text{graph}(F)$ whenever $W \in \mathcal{W}$.

Clearly, local uniform openness implies local uniform near openness. In some cases, local uniform openness of F can be derived from local uniform near openness of F (see [18, p. 145, Theorem 4], where F has a locally closed graph, and X and Y are metric spaces).

A special property of openness, which implies local uniform openness, is involved by all of the three theorems in Section 2 below: Theorem 1 (cf. [18, p. 148, Theorem 9]) shows that special property can be derived from its near version; Theorem 2 (see [19, p. 2219,Theorem 6]) shows that special property can be derived from a weaker property; Theorem 3, the main result of this paper, shows that special property can be derived from an even weaker property.

A new proof of Theorem 2 and a proof of Theorem 3 are given in Section 3. These proofs are based on some directional results presented in Sections 4 and 5. Counterexamples and some remarks are considered in Sections 6 and 7 respectively.

2. Main result

In the following, we discuss at some length the two results, Theorems 1 and 2 below, on which there is grounded our main result, Theorem 3 stated at the end of this section.

Let X and Y be metric spaces, and note F is open at (x, y) iff for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$B(y, \delta) \subseteq F(B(x, \epsilon)),$$

whereas F is nearly open at (x, y) iff for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$B(y, \delta) \subseteq \overline{F(B(x, \epsilon))}.$$

Here, $B(c, r)$ stands for *the open ball* of center c and radius r.

Further, let $\omega > 0$ be a real number, and consider the ω-openness inclusion

$$B(y, \omega\epsilon) \subseteq F(B(x, \epsilon)) \tag{3}$$

as well as the near ω-openness inclusion

$$B(y, \omega\epsilon) \subseteq \overline{F(B(x, \epsilon))}. \tag{4}$$

Theorem 1 below, a corollary of a result in [18, p. 148, Theorem 9], concludes that a special openness based on inclusion (3) can be derived from the corresponding near version, based on inclusion (4). To state this corollary, we have to particularize a construction in [18, p. 148]. For every $(x, y) \in \text{graph}(F)$, let $\eta_\omega(x, y)$ stand for the supremum of all $\epsilon > 0$ which render true the inclusion

$$(B(x, \epsilon) \times B(y, \omega\epsilon)) \cap \overline{\text{graph}(F)} \subseteq \text{graph}(F).$$

By convention, $\sup\emptyset = 0$. Note (see [19, p. 2216]) the everywhere nonnegative function $\eta_\omega : \mathrm{graph}(F) \to R\cup\{+\infty\}$ is everywhere infinite iff F has a closed graph, and everywhere finite iff F does not have a closed graph, in which case

$$|\eta_\omega(x',y') - \eta_\omega(x,y)| \le \max\{d(x',x), d(y',y)/\omega\}$$

for all $(x,y) \in \mathrm{graph}(F)$ and for all $(x',y') \in \mathrm{graph}(F)$. Note also η_ω is everywhere positive iff F has a locally closed graph.

Consider now two ϵ-plenty conditions based on inclusions (3) and (4):

- $\mathcal{A}$**:** for every $(x,y) \in \mathrm{graph}(F)$ and for every $\epsilon \in (0, \eta_\omega(x,y))$ there holds inclusion (3);
- $\mathcal{B}$**:** for every $(x,y) \in \mathrm{graph}(F)$ and for every $\epsilon \in (0, \eta_\omega(x,y))$ there holds inclusion (4).

Theorem 1 (cf. [18, p. 148, Theorem 9])**.** *Let X and Y be metric spaces, let the metric space X be complete, and let the multifunction $F : X \to Y$ have a locally closed graph. Let $\omega > 0$ be a real number. Then conditions $\mathcal{A}$ and $\mathcal{B}$ are equivalent.*

Condition $\mathcal{A}$ is the best one in that inclusion (3) may fail if $\epsilon > \eta_\omega(x,y)$. Indeed, if $\mathrm{graph}(F) = \{(x,x) \in R^2; x > 0\}$, then $\eta_1(x,x) = x$. Moreover, $B(x,\epsilon) = (x-\epsilon, x+\epsilon) \not\subseteq (0, x+\epsilon) = F(B(x,\epsilon))$ whenever $\epsilon > \eta_1(x,x)$.

If F has a closed graph, then condition $\mathcal{A}$ implies uniform openness on $\mathrm{graph}(F)$: for every $\epsilon > 0$ and for every $(x,y) \in \mathrm{graph}(F)$ there holds inclusion (3). Otherwise, it implies only local uniform openness: there exists an open covering $\mathcal{W}$ of $\mathrm{graph}(F)$ such that for every $W \in \mathcal{W}$ there exists $\zeta > 0$ such that for every $\epsilon \in (0,\zeta)$ and for every $(x,y) \in W \cap \mathrm{graph}(F)$ there holds inclusion (3). A suitable open covering $\mathcal{W}$ is the collection of sets

$$W(a,b) = B(a,\theta(a,b)) \times B(b,\omega\theta(a,b))$$

where $(a,b) \in \mathrm{graph}(F)$ and $\theta(a,b) = \eta_\omega(a,b)/2$. Moreover, for each member of the collection, a suitable ζ is $\theta(a,b)$ because

$$\theta(a,b) \le \inf\{\eta_\omega(x,y); (x,y) \in W(a,b) \cap \mathrm{graph}(F)\}.$$

Indeed, if $(a,b) \in \mathrm{graph}(F)$ and $(x,y) \in W(a,b) \cap \mathrm{graph}(F)$, then

$$|\eta_\omega(x,y) - 2\theta(a,b)| = |\eta_\omega(x,y) - \eta_\omega(a,b)| \le \max\{d(x,a), d(y,b)/\omega\} \le \theta(a,b),$$

hence $\theta(a,b) \le \eta_\omega(x,y)$.

Theorem 2 below improves on Theorem 1 by replacing the ϵ-plenty condition $\mathcal{B}$ with an ϵ-scanty condition. In the setting of the new result, the metric space Y is also complete and resembles normed spaces.

The definition of resemblance is suggested by the elementary fact that, if M is a metric space, $c \in M$, $r > 0$, and $r' > 0$, then $B(B(c,r),r') \subseteq B(c, r+r')$, and moreover, the corresponding equality holds provided that M is a normed space. Here, $B(S,r)$ stands for the union of all open balls $B(c, r)$ with $c \in S$. Note parenthetically that $B(S, r) = B(\overline{S}, r)$ whenever $S \subseteq M$ and $r > 0$.

The metric space M is said to *resemble normed spaces* if $B(B(c, r), r') = B(c, r + r')$ for all $c \in M$, $r > 0$, and $r' > 0$ (see Definition 2.2 in [17, p. 204]; cf. the definition of γ-convexity in [7, p. 271]). For further properties of metric spaces which resemble normed spaces we refer to [17, p. 9, Theorem 6.1].

Consider now an ϵ-scanty condition based on inclusion (4):

$\mathcal{C}$**:** for every $(x, y) \in \operatorname{graph}(F)$ and for every $\zeta > 0$ there exists $\epsilon \in (0, \zeta)$ such that inclusion (4) holds.

Theorem 2 (see [19, p. 2216, Theorem 6])**.** *Let X and Y be complete metric spaces, let the metric space Y resemble normed spaces, and let the multifunction $F : X \to Y$ have a locally closed graph. Let $\omega > 0$ be a real number. Then conditions $\mathcal{A}$ and $\mathcal{C}$ are equivalent.*

Further, let $\alpha \in (0, 1)$ be a real number. If $\zeta > 0$ and if for every $\epsilon \in (0, \zeta)$ there holds inclusion (4), then for every $\epsilon \in (0, \zeta)$ there holds also the inclusion

$$B(B(y, \omega(1 - \alpha)\epsilon), \omega\alpha\epsilon) \subseteq B(F(B(x, (1 - \alpha)\epsilon)), \omega\alpha\epsilon)$$

(recall a parenthetic note above). If, in addition, the metric space Y resembles normed spaces, then the preceding inclusion states

$$B(y, \omega\epsilon) \subseteq B(F(B(x, (1 - \alpha)\epsilon)), \omega\alpha\epsilon). \tag{5}$$

Let us say for the time being that a set $\Sigma \subseteq \operatorname{graph}(F) \times (0, +\infty)$ is a *pre-complete system* if for every $(x, y) \in \operatorname{graph}(F)$ there exists $\zeta > 0$ such that $\{(x, y)\} \times (0, \zeta) \subseteq \Sigma$ (cf. [5, p. 16, Definition 1.1]).

For example, if F has a locally closed graph, then the set

$$\Sigma = \{((x, y), \epsilon); (x, y) \in \operatorname{graph}(F), \epsilon \in (0, \eta_\omega(x, y))\}$$

is a pre-complete system, and condition $\mathcal{A}$ states inclusion (3) holds for all points $((x, y), \epsilon)$ of Σ, whereas condition $\mathcal{B}$ states that inclusion (4) holds for all points $(x, y), \epsilon)$ of Σ.

In the setting of some *subtraction* results, a corollary states that inclusion (3) holds for all points $((x, y), \epsilon)$ of a certain pre-complete system Σ if so does inclusion (4) (cf. [12, p. 97], where X and Y are normed spaces, and F is a linear continuous function, therefore $\Sigma = \operatorname{graph}(F) \times (0 + \infty)$; see [5, p. 17, Theorem 1.5], where F is a function; see [3, pp. 47, 48, Theorem 3.1] where Y is a normed space and F has a complete graph).

The corresponding equivalence, however, may fail if Y does not resemble normed spaces (see the second counterexample in Section 6, where $\Sigma = \operatorname{graph}(F) \times (0, +\infty)$).

If Y is complete and resembles normed spaces, we can get rid of any pre-complete system Σ as far as inclusion (5) is concerned. Actually, we can improve on condition $\mathcal{C}$ by replacing inclusion (4) with the near version of inclusion (5), namely with the inclusion

$$B(y,\omega\epsilon) \subseteq \overline{B(F(B(x,(1-\alpha)\epsilon)),\omega\alpha\epsilon)}. \tag{6}$$

(cf. [3, p. 40, Corollary 2.2]). Our main result involves an ϵ-scanty condition based on inclusion (6):

> $\mathcal{D}$**:** for every $(x,y) \in \mathrm{graph}(F)$ and for every $\zeta > 0$ there exists $\epsilon \in (0,\zeta)$ such that inclusion (6) holds.

Theorem 3. *Let X and Y be complete metric spaces, let the metric space Y resemble normed spaces, and let the multifunction $F : X \to Y$ have a locally closed graph. Let $\omega > 0$ and $\alpha \in (0,1)$ be real numbers. Then conditions $\mathcal{A}$ and $\mathcal{D}$ are equivalent.*

Theorem 3 may fail if the metric space Y does not resemble normed spaces (see the third counterexample in Section 6).

3. Proof of Theorems 2 and 3

Let $\Omega > 0$ and consider the v-*directional* inequality

$$d(v, B(y,\Omega\epsilon) \cap F(B(x,\epsilon))) < d(v,y)(1-\epsilon/\zeta). \tag{7}$$

Here, $d(p,S)$ stands for the distance from the point p to the set S. This inequality is equivalent to the v-*directional* relation

$$\emptyset \neq B(v, d(v,y)(1-\epsilon/\zeta)) \cap B(y,\Omega\epsilon) \cap F(B(x,\epsilon)), \tag{8}$$

which makes sense iff both $v \in Y \setminus \{y\}$ and $\epsilon \in (0,\zeta)$, and which is surely false if $\zeta \leq d(v,y)/\Omega$, for it is empty the intersection of the open balls $B(v, d(v,y)(1-\epsilon/\zeta))$ and $B(y,\Omega\epsilon)$.

Further, let $\omega > 0$ and $\Omega > 0$ be real numbers, and consider an ϵ-scanty condition based on inequality (7), namely the *global* condition

> $\mathcal{G}(\Omega)$**:** for every $(x,y) \in \mathrm{graph}(F)$, for every $v \in Y \setminus \{y\}$, and for every $\zeta > d(v,y)/\omega$ there exists $\epsilon \in (0,\zeta)$ such that inequality (7) holds.

Condition $\mathcal{G}(\Omega)$ is surely false if $\Omega \in (0,\omega)$, for relation (8) is surely false whenever $\zeta \in (d(v,y)/\omega, d(v,y)/\Omega]$. Obviously, if $0 < \Omega < \Omega'$, then condition $\mathcal{G}(\Omega)$ implies condition $\mathcal{G}(\Omega')$.

Each of the conditions $\mathcal{C}$ and $\mathcal{D}$ implies a condition $\mathcal{G}(\Omega)$ for some $\Omega \geq \omega$.

Proposition 1. *Let the metric space Y resemble normed spaces. Let $\omega > 0$ be a real number. Then condition $\mathcal{C}$ implies condition $\mathcal{G}(\omega)$.*

Proposition 1 is a corollary of the pointwise lemma below.

Lemma 1. *Let the metric space Y resemble normed spaces. Let $\omega > 0$ be a real number. Let $(x,y) \in \mathrm{graph}(F)$, let $v \in Y \setminus \{y\}$, let $\zeta > d(v,y)/\omega$, and let $\epsilon \in (0,\zeta)$ such that inclusion (4) holds. Let $\Omega = \omega$. Then inequality (7) holds too.*

Proof. Let $\theta = d(v,y)/(\zeta\omega)$ and note $\theta \in (0,1)$. Since inclusion (4) holds, it follows

$$B(v, d(v,y) - \theta\omega\epsilon) \cap B(y, \omega\epsilon) \subseteq \overline{B(F(B(x,\epsilon)))}.$$

Since Y resembles normed spaces, it follows the left hand side of the preceding inclusion is nonempty, and so is the set

$$B(v, d(v,y) - \theta\omega\epsilon) \cap B(y, \omega\epsilon) \cap B(F(B(x,\epsilon))).$$

Therefore inequality (7) holds. ☐

Proposition 2. *Let the metric space Y resemble normed spaces. Let $\omega > 0$ and $\alpha \in (0,1)$ be real numbers, and let $\Omega = \omega(1+\alpha)/(1-\alpha)$. Then condition $\mathcal{D}$ implies condition $\mathcal{G}(\Omega)$.*

To prove this result, we note that condition $\mathcal{D}(\alpha)$ can be rephrased as follows: for every $(x,y) \in \operatorname{graph}(F)$ and for every $\zeta > 0$ there exists $\epsilon \in (0,\zeta)$ such that

$$B\left(y, \frac{\omega}{1-\alpha}\,\epsilon\right) \subseteq \overline{B\left(F(B(x,\epsilon)), \frac{\omega\alpha}{1-\alpha}\,\epsilon\right)}. \tag{9}$$

Now, Proposition 2 is a corollary of the pointwise lemma below.

Lemma 2. *Let the metric space Y resemble normed spaces. Let $\omega > 0$ and $\alpha \in (0,1)$ be real numbers. Let $(x,y) \in \operatorname{graph}(F)$, let $v \in Y \setminus \{y\}$, let $\zeta > d(v,y)/\omega$, and let $\epsilon \in (0,\zeta)$ such that inclusion (9) holds. Let $\Omega = \omega(1+\alpha)/(1-\alpha)$. Then inequality (7) holds too.*

Proof. Let $\theta = \alpha + (1-\alpha)d(v,y)/(\omega\zeta)$, and note $\theta \in (0,1)$. Since inclusion (9) holds, it follows

$$B\left(v, d(v,y) - \theta\frac{\omega}{1-\alpha}\,\epsilon\right) \cap B\left(y, \frac{\omega}{1-\alpha}\,\epsilon\right) \subseteq \overline{B\left(F(B(x,\epsilon)), \frac{\omega\alpha}{1-\alpha}\,\epsilon\right)}.$$

Since Y resembles normed spaces, it follows open set in the left hand side of the preceding inclusion is nonempty, and so is the set

$$B\left(v, d(v,y) - \theta\frac{\omega}{1-\alpha}\,\epsilon\right) \cap B\left(y, \frac{\omega}{1-\alpha}\,\epsilon\right) \cap B\left(F(B(x,\epsilon)), \frac{\omega\alpha}{1-\alpha}\,\epsilon\right).$$

Let q be a point of the preceding set. Then there exists $q' \in F(B(x,\epsilon))$ such that

$$q' \in B\left(q, \frac{\omega\alpha}{1-\alpha}\,\epsilon\right) \subseteq B(v, d(v,y)(1-\epsilon/\zeta)) \cap B(y, \Omega\epsilon).$$

Therefore relation (8) holds. ☐

The proof of Theorem 2 follows from the sequence of implications

$$\mathcal{A} \overset{1st}{\Rightarrow} \mathcal{C} \overset{2nd}{\Rightarrow} \mathcal{G}(\omega) \overset{3rd}{\Rightarrow} \mathcal{A}.$$

The first implication is obvious. The second implication follows from Proposition 1 above. The third implication follows from Theorem 4 below.

The proof of Theorem 3 follows from the sequence of implications

$$\mathcal{A} \overset{1st}{\Rightarrow} \mathcal{D} \overset{2nd}{\Rightarrow} \mathcal{G}(\Omega) \overset{3rd}{\Rightarrow} \mathcal{A}$$

where $\Omega = \omega(1+\alpha)/(1-\alpha)$. The first implication is obvious. The second implication follows from Proposition 2 above. The third implication follows from Theorem 4 below.

Theorem 4. *Let X and Y be complete metric spaces, let the metric space Y resemble normed spaces, and let the multifunction $F : X \to Y$ have a locally closed graph. Let $\Omega \geq \omega > 0$ be real numbers. Then conditions $\mathcal{A}$ and $\mathcal{G}(\Omega)$ are equivalent.*

4. Proof of Theorem 4

Let H be the set of all positive functions $\eta : \text{graph}(F) \to R \cup \{+\infty\}$, let $\eta \in H$, consider the ϵ-plenty condition

> $\mathcal{A}(\eta)$**:** for every $(x, y) \in \text{graph}(F)$ and for every $\epsilon \in (0, \eta(x, y))$ there holds inclusion (3),

and note that, if F has a locally closed graph, then $\eta_\omega \in H$ and condition $\mathcal{A}(\eta_\omega)$ reduces to condition $\mathcal{A}$.

Proposition 3. *Let the metric space Y resemble normed spaces. Let $\omega > 0$ be a real number and let $\eta \in H$. Then condition $\mathcal{A}(\eta)$ implies condition $\mathcal{G}(\omega)$.*

Proposition 3 is a corollary of the pointwise lemma below.

Lemma 3. *Let the metric space Y resemble normed spaces, let $\Omega = \omega$, let $(x, y) \in \text{graph}(F)$, and let $v \in Y \setminus \{y\}$. If there exists $\zeta > 0$ such that for every $\epsilon \in (0, \zeta)$ there holds inclusion (3), then for every $\zeta > d(v, y)/\omega$ there exists $\epsilon \in (0, \zeta)$ such that inequality (7) holds.*

Proof. Let $\zeta' > 0$ such that condition (3) holds for all $\epsilon \in (0, \zeta')$, let $\zeta'' > d(v, y)/\omega$, and let any $\epsilon \in (0, \zeta') \cap (0, \zeta''))$. Then it is nonempty the subset $B(v, d(v, y)(1 - \epsilon/\zeta)) \cap B(y, \omega\epsilon)$ of $F(B(x, \epsilon))$, and relation (8) holds. □

Further, for every $\Omega > 0$ and for every $\eta \in H$, consider the ϵ-scanty condition

> $\mathcal{G}(\Omega, \eta)$**:** for every $(x, y) \in \text{graph}(F)$, for every $v \in B(y, \omega\eta(x, y)) \setminus \{y\}$, and for every $\zeta > d(v, y)/\omega$ there exists $\epsilon \in (0, \zeta)$ such that inequality (7) holds.

By convention, $B(y, +\infty) = Y$. Obviously, condition $\mathcal{G}(\Omega)$ implies condition $\mathcal{G}(\Omega, \eta)$ for all $\eta \in H$.

Further, for every $\Omega > 0$, denote by H_Ω the set of all $\eta \in H$ with the following properties:

$0 < \eta(x, y) \leq \eta_\Omega(x, y)$ for all $(x, y) \in \text{graph}(F)$;

η is either everywhere infinite or everywhere finite, in which case
$|\eta(x', y') - \eta(x, y)| \leq \max\{d(x', x), d(y', y)/\Omega\}$ for all $(x, y) \in \operatorname{graph}(F)$ and for all $(x', y') \in \operatorname{graph}(F)$.

Clearly, H_Ω is nonempty iff $\operatorname{graph}(F)$ is locally closed, in which case $\eta_\Omega \in H_\Omega$.

Now, the proof of Theorem 4 follows from the sequence of implications

$$\mathcal{A}(\eta_\omega) \overset{1st}{\Rightarrow} \mathcal{G}(\omega) \overset{2nd}{\Rightarrow} \mathcal{G}(\Omega) \overset{3rd}{\Rightarrow} \mathcal{G}(\Omega, \eta_\Omega) \overset{4th}{\Rightarrow} \mathcal{A}(\eta_\Omega) \overset{5th}{\Rightarrow} \mathcal{G}(\omega) \overset{6th}{\Rightarrow} \mathcal{G}(\omega, \eta_\omega) \overset{7th}{\Rightarrow} \mathcal{A}(\eta_\omega).$$

The first and fifth implications follows from Proposition 3 above. The second, third, and sixth implications are obvious. The fourth and seventh implications follows from Theorem 5 below, where the metric space Y does not necessarily resemble normed spaces.

Theorem 5. *Let X and Y be complete metric spaces, and let the multifunction F have a locally closed graph. Let $\Omega \geq \omega > 0$ be real numbers, and let $\eta \in H_\Omega$. Then conditions $\mathcal{A}(\eta)$ and $\mathcal{G}(\Omega, \eta)$ are equivalent.*

5. Proof of Theorem 5

First we rephrase condition $\mathcal{A}(\eta)$ in a form which resembles condition $\mathcal{G}(\Omega, \eta)$. The lemma below will be usefull in this regard.

Lemma 4. *Let $\omega > 0$ be a real number. Let $(x, y) \in \operatorname{graph}(F)$ and let $\zeta > 0$. Then the following two conditions are equivalent:*

for every $\epsilon \in (0, \zeta)$ there holds inclusion (3);
for every $v \in B(y, \omega\zeta) \setminus \{y\}$ and for every $\epsilon \in (d(v, y)/\omega, \zeta)$ there holds the relation $v \in F(B(x, \epsilon))$.

Proof. First, let the former condition be satisfied, let $v \in B(y, \epsilon\zeta) \setminus \{y\}$, and let $\epsilon \in (d(v, y)/\omega, \zeta)$. Since $\epsilon \in (0, \zeta)$, it follows inclusion (3) holds, hence $v \in F(B(x, \epsilon))$, and the latter condition is satisfied too.

Finally, let the latter condition be satisfied, let $\epsilon \in (0, \zeta)$, and let $v \in B(y, \omega\epsilon)$. If $v = y$, then $v \in F(x)$. If $v \neq y$, then $v \in B(y, \omega\zeta) \setminus \{y\}$ and $\epsilon \in (d(v, y)/\omega, \zeta)$, hence $v \in B(F(x, \epsilon))$. To conclude, inclusion (3) holds, and the former condition is satisfied too. □

In view of this lemma, for every $\eta \in H$, condition $\mathcal{A}(\eta)$ can be rephrased as follows:

for every $(x, y) \in \operatorname{graph}(F)$, for every $v \in B(y, \omega\eta(x, y)) \setminus \{y\}$, and for every $\epsilon \in (d(v, y)/\omega, \eta(x, y))$ there holds the relation $v \in F(B(x, \epsilon))$.

Further, for every $\eta \in H$ and $v \in Y$, consider the set

$$\mathcal{S}(\eta, v) = \{(x, y) \in \operatorname{graph}(F); v \in B(y, \omega\eta(x, y)) \setminus \{y\}\}.$$

Now, conditions $\mathcal{A}(\eta)$ and $\mathcal{G}(\Omega, \eta)$ can be rephrased as follows:

for every $v \in Y$, for every $(x, y) \in \mathcal{S}(\eta, v)$, and for every $\epsilon \in (d(v, y)/\omega, \eta(x, y))$ there holds the relation $v \in F(B(x, \epsilon)$.
for every $v \in Y$, for every $(x, y) \in \mathcal{S}(\eta, v)$, and for every $\zeta > d(v, y)/\omega$ there exists $\epsilon \in (0, \zeta)$ such that inequality (7) holds;

Therefore conditions $\mathcal{A}(\eta)$ and $\mathcal{G}(\Omega, \eta)$ are equivalent provided that so are their v-components.

Theorem 6. *Let the metric spaces X and Y be complete, and let the multifunction $F : X \to Y$ have a locally closed graph. Let $\Omega \geq \omega > 0$ be real numbers, let $\eta \in H_\Omega$, and let $v \in Y$. Then the following two conditions are equivalent:*

for every $(x, y) \in \mathcal{S}(\eta, v)$ and for every $\epsilon \in (d(v, y)/\omega, \eta(x, y))$ there holds the relation $v \in F(B(x, \epsilon))$.
for every $(x, y) \in \mathcal{S}(\eta, v)$ and for every $\zeta > d(v, y)/\omega$ there exists $\epsilon \in (0, \zeta)$ such that inequality (7) holds;

Proof. First, let the former condition be satisfied, let $(x, y) \in \mathcal{S}(\eta, v)$, let $\zeta > d(v, y)/\omega$, let any $\epsilon \in (d(v, y)/\omega, \eta(x, y)) \cap (d(v, y)/\omega, \zeta)$, and note $v \in B(v, d(v, y)(1 - \epsilon/\zeta)) \cap B(y, \Omega\epsilon)$. According to the former condition, there holds the relation $v \in F(B(x, \epsilon))$, hence relation (8) holds, and the latter condition is satisfied too.

Second, let the latter condition be satisfied, and let $(x, y) \in \mathcal{S}(\eta, v)$ and $\epsilon \in (d(v, y)/\omega, \eta(x, y))$. We have to show that $v \in F(B(x, \epsilon))$. Let $\theta \in (d(v, y)/(\omega\epsilon), 1)$. Following the spirit of some ideas in [1, p. 195], [2, p. 76], [8, p. 572], and [10, p. 30] (cf. also [14, p. 222], [15, pp. 81, 82], [16, p. 404], [17, p. 208], [19, p. 2217]), we endow the space $X \times Y$ with the metric

$$d_\Omega((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), d(y_1, y_2)/\Omega\},$$

and we apply the variational principle of Ekeland [6, p. 324] to the function

$$(p, q) \in \overline{\text{graph}(F)} \to d(v, q) \in R$$

in order to get a point $(a, b) \in \overline{\text{graph}(F)}$ such that

$$d(v, b) + \theta\omega d_\Omega((a, b), (x, y)) \leq d(v, y)$$

(see [1, p. 195] and [11, p. 815]) and such that

$$d(v, b) < d(v, q) + \theta\omega d_\Omega((p, q), (a, b))$$

for every $(p, q) \in \overline{\text{graph}(F)} \setminus \{(a, b)\}$. Since $d(v, y) < \theta\omega\epsilon$, it follows from the former inequality of the Ekeland principle that $d_\Omega((a, b), (x, y)) < \epsilon$, hence $a \in B(x, \epsilon)$ and $b \in B(y, \Omega\epsilon)$. Since $\epsilon < \eta(x, y) \leq \eta_\Omega(x, y)$, it follows

$$(B(x, \epsilon) \times B(y, \Omega\epsilon)) \cap \overline{\text{graph}(F)} \subseteq \text{graph}(F).$$

To conclude, $(a, b) \in \operatorname{graph}(F)$, hence $b \in F(a) \subseteq F(B(x, \epsilon))$. We claim that $v = b$. Suppose, to the contrary, that $b \neq v$. Since $d(v, y) < \theta\omega\eta(x, y)$, it follows from the former inequality of the Ekeland principle and from the Lipschitz property of the function $\eta \in H_\Omega$ that

$$d(v, b)+\theta\omega d_\Omega((a, b), (x, y)) < \theta\omega\eta(x, y) \leq \theta\omega\eta(a, b)+\theta\omega d_\Omega((a, b), (x, y)),$$

therefore $d(v, b) < \theta\omega\eta(a, b)$. To conclude, $(a, b) \in \mathcal{S}(\eta, v)$. Now, let $\mu = d(v, b)/(\theta\omega)$. Since $(a, b) \in \mathcal{S}(\eta, v)$ and $\mu > d(v, b)/\omega$, it follows by the later condition that there exists $\lambda \in (0, \mu)$ such that the set

$$S = B(v, d(v, b)(1 - \lambda/\mu)) \cap B(b, \Omega\lambda) \cap F(B(a, \lambda))$$

is nonempty. Let $q \in S$. Since $q \in F(B(a, \lambda))$, it follows there exists $p \in B(a, \lambda)$ such that $q \in F(p)$. Since $d(q, b) < \Omega\lambda$, it follows

$$d_\Omega((p, q), (a, b)) < \lambda.$$

Since $q \in S$, it follows $d(q, v) < d(v, b)(1-\lambda/\mu)$, hence $q \neq b$, therefore $(p, q) \in \operatorname{graph}(F)\backslash\{(a, b)\}$. Further, it follows from the latter inequality of the Ekeland principle that $d(v, b) < d(v, b)(1-\lambda/\mu)+\theta\omega\lambda$, therefore $d(v, b)/(\theta\omega) < \mu$, a contradiction. To conclude, $v = b$, the former condition is satisfied too, and the proof is accomplished. □

6. Counterexamples

The first counterexample shows that near openness at a point may fail to imply openness at that point even if the multifunction has a closed graph (cf. [19, §-11]). Let Q be the set of all rational numbers, consider the Hilbert space $l^2(Q)$, and let $F : l^2(Q) \to R$ be given through

$$\operatorname{graph}(F) = \{(q\delta_q, q); q \in Q\}.$$

Here, $\delta_q : Q \to R$ stands for the Kronecker function, i.e. $\delta_q(p) = 1$ if $p = q$, whereas $\delta_q(p) = 0$ if $p \neq q$, so that $\delta_q \in l^2(Q)$ and $\|\delta_q\| = 1$. Clearly, $(0, 0) \in \operatorname{graph}(F)$. Moreover, $F(B(0, \epsilon)) = Q \cap B(0, \epsilon)$ and $\overline{F(B(0, \epsilon))} = \overline{B(0, \epsilon))}$ for all $\epsilon > 0$, hence F is nearly open at $(0, 0)$ but F is not open there. Closeness of the graph of F follows from the fact that $\|q\delta_q - q'\delta_{q'}\| = \sqrt{|q|^2 + |q'|^2}$ whenever $q \neq q'$.

The second counterexample shows that, if the metric space Y does not resemble normed spaces, then the implication stated by the result in [5, p. 17, Theorem 1.5] can not be improved to an equivalence.

Let M be a set with at least two points, and let the set M be endowed with the discrete metric (see [4, p. 30]), i.e. $d(p, q) = 0$ if $p = q$, whereas $d(p, q) = 1$ if $p \neq q$. Clearly, M is complete. In addition, $B(c, r) = \{c\}$ if $0 < r \leq 1$, whereas $B(c, r) = M$ if $r > 1$. Accordingly, $B(B(c, r), r') = \{c\} \neq M = B(c, r+r')$ if $r \leq 1$, $r' \leq 1$, and $1 < r+r'$. To conclude, M does not resemble normed spaces.

Further, let $F : M \to M$ be the multifunction given through $\operatorname{graph}(F) = \{(x, y); y = x\}$. Then F has a closed graph.

Now, let $\omega = 1$ and let $\alpha \in (0,1)$. Then inclusion (3) holds for all $(x,y) \in \mathrm{graph}(F)$ and for all $\epsilon > 0$, but inclusion (5) does not hold, namely it does not hold for any $\epsilon \in (1, 1/\alpha] \cap (1, 1/(1-\alpha)]$ because $B(y,\omega\epsilon) = M$, whereas $B(F(B(x,(1-\alpha)\epsilon)), \omega\alpha\epsilon) = \{x\}$.

The third counterexample shows that, if the metric space Y does not resemble normed spaces, then Theorems 2 and 3 may fail. The simplest counterexample follows (cf. [17, pp. 209, 210, Counterexample 5.2]).

Indeed, let $\omega > 1$ and let $\alpha \in (0,1)$. Then condition $\mathcal{A}$ does not hold, namely inclusion (3) does not hold for any $\epsilon \in (1/\omega, 1]$, because $B(y,\omega\epsilon) = M$ but $F(B(x,\epsilon)) = \{x\}$. Nevertheless, condition $\mathcal{C}$ does hold and condition $\mathcal{D}(\alpha)$ does hold for each $\alpha \in (0,1)$, namely inclusions (4) and (5) hold for all $\epsilon \in (0, 1/\omega]$, because $B(y,\omega\epsilon)$, $\overline{F(B(x,\epsilon))}$, and $B(F(B(x,(1-\alpha)\epsilon)), \omega\alpha\epsilon)$ equal $\{x\}$.

To close this section we illustrate, in case of the multifunction $F : M \to M$ above, all the situations which involve the three items of Theorem 6, that is, the set $\mathcal{S}(\eta, v)$ and the two equivalent conditions based on the set $\mathcal{S}(\eta, v)$: the set $\mathcal{S}(\eta, v)$ is empty and the two equivalent conditions are true (but useless); the set $\mathcal{S}(\eta, v)$ is nonempty and the two equivalent conditions are false; the set $\mathcal{S}(\eta, v)$ is nonempty and the two equivalent conditions are true.

Let $\Omega \geq \omega > 0$, let $\eta \in H$, and let $v \in M$. First, $(x,y) \in \mathcal{S}(\eta, v)$ iff both $y \neq v$ and $\eta(x,y) > 1/\omega$. Second, if $(x,y) \in \mathcal{S}(\eta, v)$, then for every $\epsilon \in (d(v,y)/\omega, \eta(x,y))$ there holds the relation $v \in F(B(x,\epsilon))$ iff $(1/\omega, \eta(x,y)) \subseteq (1, +\infty)$, that is, $\omega \leq 1$. Third, if $(x,y) \in \mathcal{S}(\eta, v)$, then for every $\zeta > d(v,y)/\omega$ there exists $\epsilon \in (0,\zeta)$ such that relation (8) holds iff for every $\zeta > 1/\omega$ there holds the relation

$$\emptyset \neq (0,\zeta) \cap (1/\Omega, +\infty) \cap (1, +\infty),$$

that is, $\omega \leq 1$. Therefore: if $\eta(x,y) \leq 1/\omega$ for all $y \neq v$, then the set $\mathcal{S}(\eta, v)$ is empty; if $1/\omega < \eta(x,y)$ for some $y \neq v$, then the set $\mathcal{S}(\eta, v)$ is nonempty; if $\omega > 1$, then the two equivalent conditions are false; if $\omega \leq 1$, then the two equivalent conditions are true.

7. Final remarks

The global condition $\mathcal{G}(\Omega)$ implies the local condition

> $\mathcal{L}(\Omega)$**:** for every $(x,y) \in \mathrm{graph}(F)$ there exists $\delta > 0$ such that for every $v \in B(y,\delta) \setminus \{y\}$ and for every $\zeta > d(v,y)/\omega$ there exists $\epsilon \in (0,\zeta)$ such that inequality (7) holds.

The question arises whether the converse implication holds if the metric space Y resembles normed spaces. A partial answer is given by the result below. In the following we say a metric space M resembles normed spaces *to a greater degree* if for every $c \in M$ and for every $c' \in Y$ with $c \neq c'$ as well as for every $r > 0$ and for every $r' > 0$ with $r + r' = d(c,c')$ there exist $p \in M$ such that $d(p,c) = r$ and $d(p,c') = r'$. A counterexample in [17, pp. 213, 214] shows that a complete metric space M which resembles normed spaces may fail to resemble normed spaces to a greater degree.

Proposition 4. *Let the metric space Y resemble normed spaces to a greater degree. Let $\Omega \geq \omega > 0$ be real numbers. Then the global condition $\mathcal{G}(\Omega)$ and the local condition $\mathcal{L}(\Omega)$ are equivalent.*

Proof. Let the local condition $\mathcal{L}(\Omega)$ be satisfied, let $(x, y) \in \operatorname{graph}(F)$, let $v \in Y \setminus \{y\}$, and let $\zeta > d(v, y)/\omega$. We have to show that there exists $\epsilon \in (0, \zeta)$ such that inequality (7) holds. In view of the local condition, this is true if $v \in B(y, \delta)$. Suppose, now, $v \notin B(y, \delta)$, that is, $\delta \leq d(v, y)$. Then $\delta/\omega \leq d(v, y)/\omega < \zeta$, hence $\lambda = \delta/(\omega\zeta) < 1$. Let $\mu = 1 - \lambda$ and choose $v' \in Y$ such that $d(v', y) = \lambda d(v, y)$ and $d(v', v) = \mu d(v, y)$. Since $\delta = \lambda\omega\zeta > \lambda d(v, y) = d(v', y)$, it follows from the local condition that there exists $\epsilon \in (0, \delta/\omega)$ such that

$$\emptyset \neq B(v', d(v', y)(1 - \epsilon\omega/\delta)) \cap B(y, \Omega\epsilon) \cap F(B(x, \epsilon)).$$

Obviously, $\epsilon \in (0, \zeta)$. Now, let w be a point of the nonempty set above. We assert that $w \in B(v, d(v, y)(1 - \epsilon/\zeta))$, hence relation (8) holds. To justify our assertion, we note $d(w, v) \leq d(w, v') + d(v', v) \leq d(v', y)(1 - \epsilon\omega/\delta) + d(v', v) = \lambda d(v, y)(1 - \epsilon\omega/\delta) + \mu d(v, y) = d(v, y)(1 - \epsilon\lambda\omega/\delta) = d(v, y)(1 - \epsilon/\zeta)$. □

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