OCTAV MAYER INSTITUTE OF MATHEMATICS
ROMANIAN ACADEMY, IAŞI BRANCH
Bd. Carol I nr. 10, 700506, Iaşi, Romania

PREPRINT SERIES OF THE
OCTAV MAYER INSTITUTE OF MATHEMATICS

STABILITY IN INVERSE SOURCE PROBLEMS FOR NONLINEAR REACTION-DIFFUSION SYSTEMS

Elena-Alexandra Melnig
Octav Mayer Institute of Mathematics of the Romanian Academy, Iaşi, Romania

Nr. 10-2020

STABILITY IN INVERSE SOURCE PROBLEMS FOR NONLINEAR REACTION-DIFFUSION SYSTEMS

Elena-Alexandra Melnig

Faculty of Mathematics, University "Al. I. Cuza" Iaşi, Romania
Octav Mayer Institute of Mathematics, Romanian Academy, Iaşi Branch

Abstract. We consider coupled parabolic systems with homogeneous boundary conditions. We establish a family of L^q- Carleman inequalities, $q \in [2, \infty)$ and use them to obtain stability estimates in L^q and L^∞ norms for the sources in terms of the solution in a subdomain. We apply these estimates to reaction-diffusion systems.

Introduction

In this paper we consider systems of semilinear parabolic equations, coupled in zero order terms, and we study an inverse problem addressing the question of source estimation in L^q and L^∞ norms in terms of norms of the solution measured in a subdomain. The systems we study arise from reaction-diffusion models which are related to physical phenomena like heat transfer, population dynamics, chemical reactions. In this context the sources have positive entries and also the solutions remain in the cone of positive functions as some extra hypotheses on the nonlinear part, related to parabolic maximum principle, are assumed.

The main tool in approaching our inverse problem is a family of generalized Carleman inequalities depending on two independent positive parameters. Global Carleman estimates were established by O.Yu. Imanuvilov in the context of controllability of parabolic equations with controls distributed in subdomains (see the lecture notes by A.Fursikov, O.Yu. Imanuvilov [12] and V.Barbu [2]).

Our result has as starting point the work of O.Yu.Imanuvilov and M. Yamamoto, [13], where the authors have considered linear parabolic equations in bounded domains and established L^2 estimates for the source. In this paper we improve the result to the more general case of L^q, respectively L^∞ estimates for the source, in a linearized model, and apply these results to nonlinear models of reaction-diffusion systems. We are able to obtain a sharper source estimate, without involving the time derivative of the solution in the right side of the estimates

2010 *Mathematics Subject Classification.* 35R30, 35K10, 35K57, 93B07 .

Key words and phrases. Parabolic systems, Carleman estimates, Inverse problems, Source estimates, Reaction-diffusion systems.

The author was supported by a grant of the Ministry of Research and Innovation, CNCS - UEFISCDI, project number PN-III-P4-ID-PCE-2016-0011.

and the method uses a family of Carleman estimates with generalized weights and an argument based on the maximum principle for coupled parabolic systems.

We use the L^2 Carleman estimates as the start point to a bootstrap procedure, which leads to a corresponding class of $L^q, q \geq 2$ Carleman estimates with independent parameters and generalized weights of exponential type for nonhomogeneous parabolic systems with various homogeneous boundary conditions. The bootstrap argument is based on the regularizing effect of the heat flow in L^p spaces (see, for example,the monograph of O.A.Ladyzenskaja, V.A.Solonikov, N.N. Ural'ceva, [14]). Other results concerning Carleman inequalities in L^q norms and using bootstrap technique were established for homogeneous parabolic equations in connection to controllability problems and regularity of the controls (see V. Barbu, [3], J.-M.Coron, S.Guerrero, L.Rosier,[5], [15]). The inverse problem for the linear system with Dirichlet boundary conditions and estimates in L^q norm was investigated in [17]. However the cited reference relies more on the hypotheses on the sources considered in [13] while here some particularities appear and sharper estimates are obtained when considering sources and solutions with positive entries.

1. Preliminaries and main results

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, $\omega \subset\subset \Omega$ be an open nonempty subset of Ω, $T > 0$ and $Q = (0,T) \times \Omega$. For a given function $y : Q \to \mathbb{R}$, denote by

$$D_t y = \frac{\partial y}{\partial t}, D_i y = \frac{\partial y}{\partial x_i}, D_{ij} y = \frac{\partial^2 y}{\partial x_i \partial x_j}.$$

We denote by $W^{1,q}(\Omega), W_0^{1,q}(\Omega)$ with $q \in [1,\infty]$ the usual Sobolev spaces and by $W_q^{2,1}(Q)$ the anisotropic Sobolev space

$$W_q^{2,1}(Q) = \left\{ w \in L^q(Q) | D_t w, D^2 w \in L^q(Q) \right\},$$

with the norm $\|w\|_{W_q^{2,1}} = \|w\|_{L^q(Q)} + \|D_t w\|_{L^q(Q)} + \|D^2 w\|_{L^q(Q)}$.

In the following we will also work with vector valued functions $y = (y_1, \dots, y_n) \in [W_q^{2,1}(Q)]^n$. When denoting the norm of such functions, if there is no confusion, we will still write in a simplified manner $\|y\|_{W_q^{2,1}} := \|y\|_{[W_q^{2,1}(Q)]^n} = \sum_{i=\overline{1,n}} \|y_i\|_{W_q^{2,1}}$.

For a given Banach space X and $1 \leq p \leq +\infty$ we will use the vector valued Lebesque and Sobolev spaces $L^p(0,T;X)$ and

$$W^{1,p}(0,T;X) = \{y \in L^p(0,T;X) : y' \in L^p(0,T;X)\}.$$

Consider also the spaces

$$L^p_{loc}(0,T;X) = \left\{ y : (0,T) \to X : y \in L^p(\epsilon, T-\epsilon; X), \forall 0 < \epsilon < \frac{T}{2} \right\}$$

and

$$W^{1,p}_{loc}(0,T;X) = \left\{ y : (0,T) \to X : y \in W^{1,p}(\epsilon, T-\epsilon; X), \forall 0 < \epsilon < \frac{T}{2} \right\}.$$

For $\alpha \in (0,1)$, denote by $C^{\alpha}(\Omega)$ respectively $C^{\alpha}(Q)$ the spaces of Hölder continuous functions defined on Ω respectively Q. For $k \in \mathbb{N}$ one denotes by $C^{k+\alpha}(\Omega)$ or $C^{k+\alpha}(Q)$ the spaces of functions with k continuos derivatives which are Hölder continuous with exponent α.

We denote by $(L_i)_{i=\overline{1,n}}$ a family of n uniformly elliptic operators of second order in divergence form

$$L_i w = -\sum_{j,k=1}^{N} D_j(a_i^{jk} D_k w) \tag{1.1}$$

with coefficients $a_i^{jk} \in W^{1,\infty}(0,T;W^{1,\infty}(\Omega))$, $i = \overline{1,n}, j,k = \overline{1,N}$. Denote by $A_i = (a_i^{jk})_{j,k=\overline{1,N}}$ the matrix of coefficients in principal part which we assume satisfying the usual uniform ellipticity condition

$$\sum_{j,k=1}^{N} a_i^{jk}(t,x)\xi_j\xi_k \geq \mu|\xi|^2, \quad \forall \xi \in \mathbb{R}^N, \quad (t,x) \in Q, \quad \mu > 0, i = \overline{1,n}. \tag{1.2}$$

Consider also the first order operators (w is considered a scalar function),

$$L_i^1 w = \sum_{k=1}^{N} b_i^k D_k w, \; i = \overline{1,n}, \tag{1.3}$$

with coefficients $b_i^k \in W^{1,\infty}(0,T;L^{\infty}(\Omega))$.

We study the following reaction-diffusion system of n coupled parabolic equations

$$\begin{cases} D_t y_i - \sum\limits_{j,k=1}^{N} D_j(a_i^{jk} D_k y_i) + L_i^1 y_i + f_i(y_1, ..., y_n) = g_i, & (0,T) \times \Omega, \\ \beta_i(x)\frac{\partial y_i}{\partial n_{A_i}} + \eta_i(x) y_i = 0, & (0,T) \times \partial\Omega, \end{cases} \tag{1.4}$$

where $g_i \geq 0, i = \overline{1,n}$ are the internal sources acting in each equation of the system. In the following, when reffering to a vector function $g = (g_i)^{\top}_{i \in \overline{1,n}}$ to be positive, like $g \geq 0$, we consider the inequality satisfied on each component of the vector, $g_i \geq 0, i = \overline{1,n}$.

In the boundary conditions, we denoted by $\frac{\partial}{\partial n_{A_i}}$ the conormal derivatives, $\frac{\partial y}{\partial n_{A_i}} = \langle A_i \nabla y, n \rangle$. We impose that $\beta_i, \eta_i \in C^2(\partial\Omega)$ such that

$$\beta_i > 0 \text{ on } \partial\Omega \qquad \text{or} \qquad \beta_i \equiv 0 \text{ and } \eta_i \equiv 1 \text{ on } \partial\Omega. \tag{1.5}$$

The first of the above boundary condition covers the Neumann and Robin type and the last one gives the Dirichlet boundary conditions.

The coupling is given through the C^1 nonlinearities $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ with $f_i(0) = 0$, $i = \overline{1,n}$ and we introduce the following hypotheses:

(H1) (*quasimonotonicity*) for some $\varepsilon_0 > 0$, $\frac{\partial f_i}{\partial y_j}(y_1, \ldots, y_n) \leq 0$, $y \in \mathcal{V}_{\varepsilon_0}(0) := \{y \geq 0,\ \|y\| \leq \varepsilon_0\}, j \neq i, i, j = 1, \ldots, n$;

(H2) $f_i(y_1, \ldots, y_{i-1}, 0, y_{i+1}, \ldots, y_n) \leq 0, i = \overline{1,n}, y \geq 0$.

In the following we consider a fixed instant of time $\theta \in (0, T)$ which can be chosen, for the ease of computations $\theta = \frac{T}{2}$.

In order to describe the framework of our problem, we introduce the following sets of functions (sources and corresponding solutions).

Let $\tilde{G}$ be a compact subset of $[L^{q'}(Q)]^n$ with $q' = \frac{q}{q-1}$ such that $0 \notin \tilde{G}$. For $q \geq 2, \tilde{c} > 0$, $\tilde{\delta} > 0$ consider the sets of sources:

$$\mathcal{G}_{q,\tilde{\delta},\tilde{G}} = \left\{ \begin{array}{c} g \in W^{1,1}((0,T); [L^q(\Omega)]^n) : \ g \geq 0 \\ \text{and } \exists \tilde{g} \in \tilde{G} \text{ s.t. } \int_Q g \cdot \tilde{g} dx dt \geq \tilde{\delta} \|g\|_{L^q(Q)} \end{array} \right\} \tag{1.6}$$

and

$$\mathcal{G}_{q,\tilde{c},\tilde{\delta},\tilde{G}} = \left\{ \begin{array}{c} g \in W^{1,1}((0,T); [L^q(\Omega)]^n) : \ g \geq 0, \\ \left| \frac{\partial g(t,x)}{\partial t} \right| \leq \tilde{c} |g(\theta, x)|, \quad \text{a.e. } (t,x) \in (0,T) \times \Omega \\ \text{and } \exists \tilde{g} \in \tilde{G} \text{ s.t. } \int_Q g \cdot \tilde{g} dx dt \geq \tilde{\delta} \|g\|_{L^q(Q)} \end{array} \right\}. \tag{1.7}$$

One may observe that the functions in set of sources (1.6) do not have the boundedness condition on the time derivative in relation to a certain observation instant $\theta \in (0, T)$.

Also, consider the set of functions,

$$\mathcal{F}_{q,M} = \{y \in [W_q^{2,1}(Q) \cap L^\infty(Q)]^n : y \geq 0,\ \|y\|_{L^\infty(Q)} \leq M\}. \tag{1.8}$$

Problem: *Obtain estimates for the sources in a reaction-diffusion system in terms of the solution in a subdomain, estimates which would guarantee that small variations of the solution observed on a subdomain correspond to small variations of the source in the whole domain.*

Remark 1. *There exist difficulties encountered when trying to extend the approach in* [13] *to L^q setting and to the nonlinear case. This lead us to introduce new classes of sources, not too restrictive for which one is able to prove stability estimates. With respect to* [13]*, thye class of sources $\mathcal{G}_{q,\tilde{c},\tilde{\delta},\tilde{g}}$ we use in this paper need to have an additional property*

$$\int_Q g \cdot \tilde{g} \geq \tilde{\delta} \|g\|_{L^q(Q)}. \tag{1.9}$$

for some $\tilde{g}$ belonging to a compact subset of $\tilde{G} \subset L^{q'}$. Observe that such a property is verified by the sources which are concentrated, in the *following sense: for $\epsilon > 0$ and $\delta_1 > 0$ fixed, for each entry i there exists an open subset $\tilde{\omega}_i \subset Q$ with Lebesque measure $\mu(\tilde{\omega}_i) > \epsilon$ such that*

$$\delta_1 \|g_i\|_{L^\infty(Q)} \leq \inf_{\tilde{\omega}_i} g_i. \tag{1.10}$$

Remark that condition (1.10) implies (1.9), by taking $\tilde{g} = (1, \ldots, 1)^\top$ and for some $\tilde{\delta} = \tilde{\delta}(\epsilon, \delta_1) > 0$.

The main results concerning the stability for nonlinear parabolic systems are the following two theorems:

Theorem 1. *(L^q stability estimates)*
Let $2 \leq q < \infty$. Let $\tilde{\delta} > 0, M > 0$, a compact set $\tilde{G} \subset L^{q'}(Q)$, $0 \notin \tilde{G}$ and assume that the sources in (1.4) belong to $\mathcal{G}_{q,\tilde{\delta},\tilde{G}}$ and the associated solutions satisfy $y \in \mathcal{F}_{q,M}$.
Assume also that one of the following conditions, (A) or (B), concerning nonlinearity f, holds :

(A) *f satisfies the hypothesis $(H1)$ in the whole cone $y \geq 0$,*

or

(B) *$q > \frac{N+2}{2}$ and f satisfies hypotheses $(H1)$, $(H2)$.*

Then an L^q stability estimate holds: there exists $C = C(\tilde{\delta}, M, \tilde{G}) > 0$ such that

$$\|g\|_{L^q(Q)} \leq C\|y\|_{L^q(Q_\omega)}. \tag{1.11}$$

Theorem 2. *(L^∞ stability estimates) Let $\alpha \in (0,1)$, $q = \frac{N+1}{1-\alpha}$ and $\theta \in (0,T)$ an intermediate observation instant of time. Consider $\tilde{\delta} > 0$, $M > 0$ and a compact set $\tilde{G} \subset L^{q'}(Q)$, $0 \notin \tilde{G}$ such that the sources in (1.4) belong to $\mathcal{G}_{q,\tilde{c},\tilde{\delta},\tilde{G}} \cap C^\alpha(Q)$ and the associated solutions $y \in \mathcal{F}_{q,M}$. Assume also that one of the conditions (A) or (B) holds.*

Then there exists $C = C(\alpha, \tilde{c}, \tilde{\delta}, M) > 0$ such that an L^∞ source estimate holds:

$$\|g\|_{L^\infty(Q)} \leq C(\|y\|_{L^q(Q_\omega)} + \|y(\theta, \cdot)\|_{C^{2+\alpha}(\Omega)}). \tag{1.12}$$

Remark 2. *We point out the fact that in the above Theorems we are not interested in the existence of solutions to Cauchy problem associated to (1.4). In the theory of reaction-diffusion processes most of the mathematical models contain nonlinear couplings of the equations and the couplings may have polynomial behaviour at infinity. Existence of global solutions is proved by specific methods. We refer, for example, to the papers [8, 9, 7, 10] for models of reaction-diffusion systems without a source, where the proof of the existence is based on the study of some entropy functional. There are also models of reaction-diffusion systems with sources playing the role of a distributed control [5, 6], where existence is proved on the given interval of time locally around a reference solution, in spaces of regular enough functions.*

In this paper we study source estimates for reaction-diffusion systems assuming that solutions exist and satisfy an a priori boundedness estimate (1.8)*.*
We refer to the monograph of M.Choulli [4] *for an introduction to inverse problems.*

The approach for obtaining source estimates for nonlinear systems is combining a priori estimates for the solution with source estimates for associated linear systems which in a certain sense approximate the nonlinear model. The results in the linear case give informations on the source in the nonlinear problem under apriori L^∞ bounds of the solutions.

Consequently, for the beginning we consider a generic linear parabolic problem, with the same principal part as the nonlinear system, with one of the homogeneous boundary conditions (Dirichlet, Neumann or Robin) on each component of the vector solution (1.4).

$$\left\{\begin{array}{ll} D_t y_i + L_i y_i + L_i^1 y_i + L_i^0 y = g_i, & (0,T)\times\Omega, \\ \beta_i(x)\frac{\partial y_i}{\partial n_{A_i}} + \eta_i(x) y_i = 0, & (0,T)\times\partial\Omega, \end{array}\right. i=\overline{1,n} \tag{1.13}$$

where $g_i \geq 0, i=\overline{1,n}$ are the internal sources and β_i, η_i are given as before in (1.5).

The lower-order operators are given by (w is a scalar function, y is vector valued function):

$$L_i^1 w = \sum_{k=\overline{1,N}} b_i^k D_k w, \quad L_i^0 y = \sum_{l=\overline{1,n}} c_i^l y_l, \quad i=\overline{1,n}, \tag{1.14}$$

with coefficients $b_i^k, c_i^l \in W^{1,\infty}(0,T;L^\infty(\Omega))$, and the coupling is done only through the zero-order terms.

We are interested in obtaining L^q and L^∞ estimates for the source $g=(g_i)_{i=\overline{1,n}} \in \mathcal{G}_{q,\tilde{\delta},\tilde{G}}$ in terms of the solution y measured in Q_ω. The result in the linear case is the following:

Theorem 3. *Let $2\leq q<\infty, \tilde{\delta}>0, \tilde{c}>0$ and a compact set $\tilde{G}\subset L^{q'}(Q)$, $0\notin\tilde{G}$. Then, for sources g in $\mathcal{G}_{q,\tilde{\delta},\tilde{G}}$ and corresponding solutions y to* (1.13) *belonging to $[W_q^{2,1}(Q)]^n$, there exists $C=C(\tilde{\delta},q)>0$, such that*

$$\|g\|_{L^q(Q)} \leq C\|y\|_{L^q(Q_\omega)}. \tag{1.15}$$

Moreover, for $\theta\in(0,T), \alpha\in(0,1)$ and sources g in $\mathcal{G}_{q,\tilde{\delta},\tilde{c},\tilde{G}}\cap C^\alpha(Q)$ with corresponding solutions y to (1.13) *belonging to $[W_q^{2,1}(Q)]^n$, there exists $C=C(\tilde{\delta},q)>0$, such that*

$$\|g\|_{L^\infty(Q)} \leq C(\|y\|_{L^q(Q_\omega)} + \|y(\theta,\cdot)\|_{C^{2+\alpha}(\Omega)}). \tag{1.16}$$

The proof of the above theorem relies on L^q Carleman estimates for the parabolic systems under homogeneous boundary conditions

(Dirichlet, Neumann or Robin) and an argument based on the Maximum Principle for systems of parabolic equations.

Concerning the Maximum Principle for single parabolic equations the results are classical and we refer to the monograph of M.H.Protter and H.F.Weinberger [18]. The case of systems of linear parabolic equations which are weakly coupled (*i.e.* coupled in zero order terms) is also treated in [18], [16] and this is the result we need and we discuss further.

We mention however that a large interest in literature is devoted to Maximum Principles for semilinear parabolic systems, formulated in terms of invariant sets. We refer to [1], [11], [19] where invariance of closed convex sets is obtained through tangency conditions and subquadratic growth for the couplings of the first order derivatives.

Consider now weakly coupled linear systems of form (1.13) where the boundary operator is given by

$$\mathcal{B}y = (B_i y_i)_{i=\overline{1,n}}, \mathcal{B}_i y_i = \beta_i(x)\frac{\partial y_i}{\partial n_{A_i}} + \eta_i(x)y_i, i = \overline{1,n}.$$

Under the additional hypothesis that the off-diagonal terms of the matrix L^0 are nonpositive,

$$c_i^l \leq 0,\ i \neq l,\ i,l \in \overline{1,n}, \tag{1.17}$$

the results from [18],[1] give that if $y_i(0,\cdot) \geq 0$ in Ω then we have $y_i \geq 0$ in the whole domain $(0,T)\times\Omega$. Moreover, if the solution is zero at an interior point $(t_0,x_0) \in (0,T)\times\Omega$ then $y \equiv 0$ for all $t < t_0$.

The main result concerning the L^q Carleman estimates for systems of linear parabolic equations (1.13), that we prove in §2 uses some auxiliary functions. Consider an open subset $\omega \subset\subset \Omega$ and a function $\psi \in C^2(\overline{\Omega})$ such that

$$\frac{1}{3} \leq \psi \leq \frac{4}{3}, \quad \psi|_{\partial\Omega} = \frac{1}{3}, \quad \{x \in \overline{\Omega} : |\nabla\psi(x)| = 0\} \subset\subset \omega.$$

One also considers the weight functions

$$\varphi(t,x) := \frac{e^{\lambda\psi(x)}}{t(T-t)}, \quad \alpha(t,x) := \frac{e^{\lambda\psi(x)} - e^{1.5\lambda\|\psi\|_{C(\overline{\Omega})}}}{t(T-t)}. \tag{1.18}$$

The result concerning the L^q Carleman estimates for systems of linear parabolic equations (1.13) is the following

Proposition 1. *(L^q-Carleman estimate) Let $g \in (L^q(Q))^n$, with $2 \leq q < \infty$. Then there exist $s_0 = s_0(q)$, $\lambda_0 = \lambda_0(q)$, such that if $\lambda > \lambda_0$, $s', s > s_0$, $\frac{s'}{s} > \Gamma > 1$, then there exists $C = C(q,\Gamma)$ such that the solutions $y \in W_q^{2,1}(Q)$ to (1.13), satisfy the estimate:*

(1.19)
$$\|ye^{s'\alpha}\|_{L^q(Q)} + \|(Dy)e^{s'\alpha}\|_{L^q(Q)} + \|(D^2y)e^{s'\alpha}\|_{L^q(Q)} + \|(D_ty)e^{s'\alpha}\|_{L^q(Q)}$$
$$\leq C\left[\|ge^{s\alpha}\|_{L^q(Q)} + \|ye^{s\alpha}\|_{L^2(Q_\omega)}\right].$$

The above result is based on the regularizing effect of the parabolic flow combined with a bootstrap argument applied to the linear parabolic system and using as a start point the following classical L^2 Carleman estimate (see [13]):

Proposition 2. *For $g \in L^2(Q)$, there exist constants $\lambda_0 = \lambda_0(\Omega, \omega)$, $s_0 = s_0(\Omega, \omega)$ such that, for any $\lambda \geq \lambda_0$, $s \geq s_0$ and some $C = C(T, \Omega, \omega)$, the following inequality holds:*
(1.20)
$$\int_Q \left[(s\varphi)^{-1}\left(|D_ty|^2 + |D^2y|^2\right) + s\lambda^2\varphi|Dy|^2 + s^3\lambda^4\varphi^3|y|^2\right] e^{2s\alpha}dxdt$$
$$\leq C\left(\int_Q |g|^2 e^{2s\alpha}dxdt + \int_{[0,T]\times\omega} s^3\lambda^4\varphi^3|y|^2e^{2s\alpha}dxdt\right),$$
for $y \in [H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))]^n$ solutions of (1.13).

2. L^q-Carleman estimates with general weights.Proof of Proposition 1

We consider the system (1.13) written in a more compact way,

$$(2.1) \qquad \begin{cases} D_ty + Ly + L^1y + L^0y = g \geq 0, (0,T)\times\Omega, \\ \mathcal{B}y = 0,\ (0,T)\times\partial\Omega. \end{cases}$$

The proof of these L^q estimates is based on the regularising properties of parabolic flows and Sobolev embeddings for anisotropic Sobolev spaces. Such classical embedding results may be found in [14], Lemma 3.3; we will use the following reduced form of the cited result:

Lemma 1. *Consider $u \in W^{2,1}_p(Q)$.*
Then $u \in Z_1$ where
$$Z_1 = \begin{cases} L^q(Q) & \text{with } q \leq \frac{(n+2)p}{n+2-2p} & \text{when } p < \frac{N+2}{2} \\ L^q(Q) & \text{with } q \in [1,\infty), & \text{when } p = \frac{N+2}{2} \\ C^{\alpha,\alpha/2}(Q) & \text{with } 0 < \alpha < 2 - \frac{N+2}{p}, & \text{when } p > \frac{N+2}{2} \end{cases}$$
and there exists $C = C(Q,p,N)$ such that
$$\|u\|_{Z_1} \leq C\|u\|_{W^{2,1}_p(Q)}.$$
Moreover, $Du \in Z_2$ where
$$Z_2 = \begin{cases} L^q(Q) & \text{with } q \leq \frac{(N+2)p}{N+2-p} & \text{when } p < N+2 \\ L^q(Q) & \text{with } q \in [1,\infty), & \text{when } p = N+2 \\ C^{\alpha,\alpha/2}(Q) & \text{with } 0 < \alpha < 1 - \frac{N+2}{p}, & \text{when } p > N+2 \end{cases}$$

and there exists $C = C(p, N)$ such that

$$\|Du\|_{Z_2} \leq C\|u\|_{W^{2,1}_p(Q)}.$$

In order to prove the L^q Carleman estimates, we introduce the following auxiliary functions based on the functions from (1.18):

$$\underline{\psi} := \inf_{x\in\Omega} \psi(x), \quad \overline{\psi} := \sup_{x\in\Omega} \psi(x),$$

$$\overline{\varphi} := \frac{e^{\lambda\overline{\psi}}}{t(T-t)}, \underline{\varphi} := \frac{e^{\lambda\underline{\psi}}}{t(T-t)}, \overline{\alpha} := \frac{e^{\lambda\overline{\psi}} - e^{1.5\lambda\overline{\psi}}}{t(T-t)}, \underline{\alpha} := \frac{e^{\lambda\underline{\psi}} - e^{1.5\lambda\overline{\psi}}}{t(T-t)}.$$

Remark 3. *Concerning the weights involved in the Carleman inequalities, one may observe that for all $m > 0$ and $\sigma_0 > 1$, there exist $\tilde{\lambda}_0 = \tilde{\lambda}_0(\sigma_0) > 0$ and $C = C(m)$ such that if $\lambda > \lambda_0$ and $s_1, s_2 > 0$ with $\frac{s_2}{s_1} = \sigma > \sigma_0$, one has*

$$\varphi^m s_2^m \lambda^m e^{s_2\alpha} \leq C(m)e^{s_1\alpha}, \tag{2.2}$$

with φ, α defined in (1.18).

It follows from the following stronger inequality:

$$\overline{\varphi}^m s_2^m \lambda^m e^{s_2\overline{\alpha}} \leq C(m)e^{s_1\underline{\alpha}}. \tag{2.3}$$

Indeed, for some $\sigma > \sigma_0$ and for some $\tilde{\lambda}_0(\sigma_0) > 0$ we have for $\lambda > \tilde{\lambda}_0$ that

$$-(\sigma-1)e^{1.5\lambda\overline{\psi}} + \sigma e^{\lambda\overline{\psi}} - e^{\lambda\underline{\psi}} \leq -\sigma\lambda e^{\lambda\overline{\psi}}.$$

Consequently, (2.3) *follows from the inequalities*

$$\frac{e^{m\lambda\overline{\psi}}\sigma^m s_1^m \lambda^m}{t^m(T-t)^m} e^{\frac{-(\sigma-1)s_1 e^{1.5\lambda\overline{\psi}} + \sigma s_1 e^{\lambda\overline{\psi}} - s_1 e^{\lambda\underline{\psi}}}{t(T-t)}} \leq \frac{e^{m\lambda\overline{\psi}}\sigma^m s_1^m \lambda^m}{t^m(T-t)^m} e^{\frac{-\sigma s_1 \lambda e^{\lambda\overline{\psi}}}{t(T-t)}}$$

$$\leq sup_{\mu\in[0,\infty)} \mu^m e^{-\mu} = C(m).$$

We proceed now to the bootstrap argument. For a given $\gamma > 1$ and $j \in \mathbb{N}$ we denote by

$$z^j := ye^{\gamma^j s\overline{\alpha}} = z^{j-1}e^{\gamma^{j-1}s\overline{\alpha}(\gamma-1)}.$$

Observe that since $\gamma > 1$, for fixed j there exists $\bar{\lambda}_0(j) > 0$ such that

$$e^{\gamma^j s\alpha} < e^{\gamma^j s\overline{\alpha}} < e^{s\alpha} \text{ for all } \lambda \geq \bar{\lambda}_0(j). \tag{2.4}$$

Each z^j verifies the problem

$$\begin{cases} D_t z^j + Lz^j = ge^{\gamma^j s\overline{\alpha}} + O[s\gamma^j\overline{\varphi}^2 e^{\gamma^{j-1}s\overline{\alpha}(\gamma-1)}]z^{j-1}, \text{ in } (0,T)\times\Omega \\ \mathcal{B}z^j = 0, \text{ on } (0,T)\times\partial\Omega, \\ z^j(0,\cdot) = 0 \text{ in } \Omega \end{cases} \tag{2.5}$$

At this point we construct a sequence $\{q_j\}_{j\in\mathbb{N}}$ taking into account the regularity argument from Lemma 1:

$$q_0 = 2, \quad q_j := \begin{cases} \dfrac{(N+2)q_{j-1}}{N+2-q_{j-1}}, \text{ if } q_{j-1} < N+2, \\ \frac{3}{2}q_{j-1}, \text{ if } q_{j-1} \geq N+2. \end{cases} \tag{2.6}$$

The sequence $\{q_j\}_{j\in\mathbb{N}}$ is increasing to infinity. Since $g \in L^q(Q)$, we consider m such that $q_{m-1} \leq q < q_m$. Observe that by standard Sobolev embedding we have:

$$W^{1,q_{j-1}}(Q) \subset L^{q_j}(Q),\ j = 1, \dots, m, \tag{2.7}$$

since the Sobolev exponent $q_j^* := \frac{(N+1)q_{j-1}}{N+1-q_{j-1}} > q_j$.

One may observe that a similar argument as in Remark 3 give that there exists $S_0, \Lambda_0 \geq 0$ and $C = C(j) > 0$ such that for $s \geq S_0, \lambda \geq \Lambda_0$ and $j = \overline{1,m}$ we have

$$s\gamma^j \overline{\varphi}^2 e^{\gamma^{j-1}s\overline{\alpha}(\gamma-1)} \leq C(j). \tag{2.8}$$

Since $z^j(0,\cdot) = 0$, from parabolic regularity and using the previous estimate (2.8), for λ big enough ($\lambda > \max\{\lambda_0, \Lambda_0, \max_{j=\overline{1,m}}\{\tilde{\lambda}_j, \bar{\lambda}_j\}\}$) we have

$$\|z^j\|_{W^{2,1}_{q_{j-1}}(Q)} \leq C\left(\|ge^{\gamma^j s\overline{\alpha}}\|_{L^{q_{j-1}}(Q)} + \|z^{j-1}\|_{L^{q_{j-1}}(Q)}\right) \tag{2.9}$$

By Lemma 1 and (2.7) we have that $z^j \in L^{q_j}(Q)$ and the previous inequality gives for $j = 1, \dots, m$

$$\|z^j\|_{L^{q_j}(Q)} \leq C\left(\|ge^{\gamma^j s\overline{\alpha}}\|_{L^{q_{j-1}}(Q)} + \|z^{j-1}\|_{L^{q_{j-1}}(Q)}\right), \tag{2.10}$$

and so

$$\|z^m\|_{L^{q_m}(Q)} \leq C\left(\sum_{j=1}^{m-1} \|ge^{\gamma^j s\overline{\alpha}}\|_{L^{q_j}(Q)} + \|z^0\|_{L^{q_0}(Q)}\right). \tag{2.11}$$

Lemma 1 gives also that $\|Dz^m\|_{L^{q_m}(Q)} \leq \|z^m\|_{W^{2,1}_{q_{m-1}}(Q)}$, meaning that we have estimates also for first order derivatives

$$\|z^m\|_{L^{q_m}(Q)} + \|Dz^m\|_{L^{q_m}(Q)} \leq C\left(\sum_{j=1}^{m-1} \|ge^{\gamma^j s\overline{\alpha}}\|_{L^{q_j}(Q)} + \|z^0\|_{L^{q_0}(Q)}\right). \tag{2.12}$$

Since

$$\|ye^{\gamma^m s\alpha}\|_{L^q(Q)} \leq \|ye^{\gamma^m s\overline{\alpha}}\|_{L^q(Q)} \leq C\|ye^{\gamma^m s\overline{\alpha}}\|_{L^{q^m}(Q)}$$

and

$$\|Dye^{\gamma^m s\alpha}\|_{L^q(Q)} \leq \|Dye^{\gamma^m s\overline{\alpha}}\|_{L^q(Q)} \leq C\|Dye^{\gamma^m s\overline{\alpha}}\|_{L^{q^m}(Q)},$$

using (2.12) we can obtain a partial estimate,

$$\|ye^{\gamma^m s\alpha}\|_{L^q(Q)}+\|Dye^{\gamma^m s\alpha}\|_{L^q(Q)} \le C\left[\sum_{j=1}^{m-1}\|ge^{\gamma^j s\overline{\alpha}}\|_{L^{q_j}(Q)}+\|z^0\|_{L^{q_0}(Q)}\right]. \tag{2.13}$$

Now we use (2.4), the fact that $q_0 = 2$, $z^0 = ye^{s\overline{\alpha}}$, and the L^2 Carleman inequality (1.20) to properly bound $\|z^0\|_{L^{q_0}(Q)}$:

$$\begin{aligned} \|ye^{s\overline{\alpha}}\|_{L^2(Q)} &\le C\|ye^{\frac{1}{\gamma}s\alpha}\|_{L^2(Q)} \\ &\le C\left(\|ge^{\frac{1}{\gamma}s\alpha}\|_{L^2(Q)}+\|s^{\frac{3}{2}}\lambda^2\varphi^{\frac{3}{2}}ye^{\frac{1}{\gamma}s\alpha}\|_{L^2(Q_\omega)}\right). \end{aligned} \tag{2.14}$$

Using (2.4), Remark 3 and the fact that $q > q_j$ for all $q_j \in \overline{1, m-1}$, there exists $C > 0$ such that the right hand-side of (2.13) is bounded as follows

$$\begin{aligned} &\sum_{j=1}^{m-1}\|ge^{\gamma^j s\overline{\alpha}}\|_{L^{q_j}(Q)}+\|ge^{\frac{1}{\gamma}s\alpha}\|_{L^2(Q)}+\|s^{\frac{3}{2}}\lambda^2\varphi^{\frac{3}{2}}ye^{\frac{1}{\gamma}s\alpha}\|_{L^2(Q_\omega)} \\ &\le C\left(\|ge^{\frac{1}{\gamma}s\alpha}\|_{L^q(Q)}+\|ye^{\frac{1}{\gamma^2}s\alpha}\|_{L^2(Q_\omega)}\right). \end{aligned} \tag{2.15}$$

In order to obtain estimates for time derivatives and second order space derivatives, we return to the parabolic problem verified by z^{m+1}:

$$\begin{cases} D_t z^{m+1} + Lz^{m+1} = ge^{\gamma^{m+1}s\overline{\alpha}} + O[s\gamma^{m+1}\overline{\varphi}^2 e^{\gamma^m s\overline{\alpha}(\gamma-1)}]z^m, \ (0,T)\times\Omega \\ \mathcal{B}z^{m+1} = 0, \ (0,T)\times\partial\Omega, \\ z^{m+1}(0,\cdot) = 0, \ \Omega \end{cases} \tag{2.16}$$

Parabolic regularity along with (2.8) gives

$$\|z^{m+1}\|_{W_q^{2,1}(Q)} \le C\left(\|ge^{\gamma^{m+1}s\overline{\alpha}(\gamma-1)}\|_{L^q(Q)}+\|z^m\|_{L^q(Q)}\right) \tag{2.17}$$

which implies, using again (2.4), (2.11) and (2.15) that

$$\begin{aligned} &\|D^2ye^{\gamma^{m+1}s\alpha}\|_{L^q(Q)} \le \|D^2ye^{\gamma^{m+1}s\overline{\alpha}}\|_{L^q(Q)} = \|D^2z^{m+1}\|_{L^q(Q)} \\ &\le C\left(\|ge^{\frac{1}{\gamma}s\alpha}\|_{L^q(Q)}+\|ye^{\frac{1}{\gamma^2}s\alpha}\|_{L^2(Q_\omega)}\right) \end{aligned} \tag{2.18}$$

and using (2.8), (2.4), (2.11), (2.15) that

$$\begin{aligned} &\|(D_ty)e^{\gamma^{m+1}s\alpha}\|_{L^q(Q)} \le \|D_tz^{m+1}\|_{L^q(Q)}+C\|z^m\|_{L^q(Q)} \\ &\le C\left(\|ge^{\frac{1}{\gamma}s\alpha}\|_{L^q(Q)}+\|ye^{\frac{1}{\gamma^2}s\alpha}\|_{L^2(Q_\omega)}\right). \end{aligned} \tag{2.19}$$

From (2.13),(2.15),(2.18) and (2.19) and taking $\gamma = \Gamma^{\frac{1}{m+3}}$, s changed into $\frac{1}{\gamma^2}s$, we obtain the conclusion,

$$\begin{aligned} &\|ye^{s'\alpha}\|_{L^q(Q)}+\|(Dy)e^{s'\alpha}\|_{L^q(Q)}+\|(D^2y)e^{s'\alpha}\|_{L^q(Q)}+\|(D_ty)e^{s'\alpha}\|_{L^q(Q)} \\ &\le C\left(\|ge^{s\alpha}\|_{L^q(Q)}+\|ye^{s\alpha}\|_{L^2(Q_\omega)}\right). \end{aligned} \tag{2.20}$$

3. L^q ESTIMATES FOR THE SOURCE TERM OF THE LINEAR SYSTEM

Lemma 2. *Let $\omega \subset\subset \Omega$, and consider $(y^m)_m$ a sequence of solutions for the system* (1.13) *with corresponding sources $(g^m)_m \subset L^q(Q)$. If $(g^m)_m$ is bounded in $L^q(Q)$ and $(y^m)_m$ is bounded in $L^q(Q_\omega)$, then there exist subsequences also denoted by $(y^m)_m$, $(g^m)_m$, and, correspondingly, $y \in L^q_{loc}(0,T;W^{1,q}(\Omega))$, $g \in L^q(Q)$ such that $y^m \longrightarrow y$ strongly in $L^q(\epsilon, T-\epsilon; W^{1,q}(\Omega))$, for all $0 < \epsilon < \frac{T}{2}$, $g^m \rightharpoonup g$ weakly $L^q(Q)$ and y is the solution to* (1.13) *corresponding to g.*

Proof. The boundedness of $(g^m)_m$ in $L^q(Q)$ and of $(y^m)_m$ in $L^q(Q_\omega)$ assures, by Carleman estimate for the solution y^m of (1.13) corresponding to the sources g^m, that $(y^m)_m$ is bounded in $W^{2,1}_q(Q^\epsilon)$, $\epsilon > 0$ arbitrarily small, where Q^ϵ is a cylinder of form $Q^\epsilon = (\epsilon, T-\epsilon) \times \Omega$. This implies that $(y^m)_m$ is bounded in $L^q(\epsilon, T-\epsilon; W^{2,q}(\Omega))$ and $D_t y^m$ is bounded in $L^q(\epsilon, T-\epsilon; L^q(\Omega))$. Since $W^{2,q}(\Omega) \subset W^{1,q}(\Omega) \subset L^q(\Omega)$ with compact embeddings, by Aubin-Lions lemma applied to sequence $(y^m)_m$, there exists $y \in L^q_{loc}(0,T;W^{1,q}(\Omega))$ and a subsequence also denoted $(y^m)_m$ such that:

$$y^m \longrightarrow y \text{ in } L^q(\epsilon, T-\epsilon; W^{1,q}(\Omega)) \text{ as } m \longrightarrow \infty, \forall \epsilon > 0. \tag{3.1}$$

Strong convergence of $(y^m)_m$ in $L^q(\epsilon, T-\epsilon; W^{1,q}(\Omega))$ and weak convergence of $(g^m)_m$ in $L^q(Q)$ allow to pass to the limit in a weak formulation of problem

$$\begin{cases} D_t y^m + L y^m + L^1 y^m + L^0 y^m = g^m, & \text{in } (0,T) \times \Omega, \\ \mathcal{B} y^m = 0, & \text{on } (0,T) \times \partial\Omega, \end{cases} \tag{3.2}$$

to conclude that y is solution to (1.13) corresponding to g. ☐

We focus now on proving source estimates for the linear parabolic systems (1.13) with sources belonging to $\mathcal{G}_{q,\tilde\delta,\tilde G}$. We have to prove that there exists $C = C(q, \tilde\delta, \tilde G)$ such that for $g \in \mathcal{G}_{q,\tilde\delta,\tilde G}$,

$$\|g\|_{L^q(Q)} \leq C \|y\|_{L^q(Q_\omega)}. \tag{3.3}$$

We argue by contradiction. Suppose it were not true. Then there exists a sequence $(g^m)_m \subset \mathcal{G}_{q,\tilde\delta,\tilde G}$ and corresponding solutions y^m such that

$$\|g^m\|_{L^q(Q)} > m \|y^m\|_{L^q(Q_\omega)}. \tag{3.4}$$

With no loss of generality we may suppose that $\|g^m\|_{L^q(Q)} = 1$ and thus, up to a subsequence, $g^m \rightharpoonup g$ weakly $L^q(Q)$, for some $g \in L^q(Q)$.

The above sequence of inequalities implies that:

$$y^m \to 0 \text{ in } L^q(Q_\omega) \text{ as } m \to \infty \text{ and } \|y^m\|_{L^q(Q_\omega)} \text{ bounded.} \tag{3.5}$$

We observe now that the weak limit g of $(g^m)_m$ is not zero. Indeed, by hypothesis $g^m \in \mathcal{G}_{q,\tilde\delta,\tilde G}$ and thus there exist a corresponding $\tilde g^m \in \tilde G$

such that
$$\int_Q g^m \tilde{g}^m \geq \tilde{\delta} \|g^m\|_{L^q(Q)} = \tilde{\delta}.$$
By further extracting a subsequence we may suppose that $\tilde{g}_m \to \tilde{g} \in \tilde{G}, \tilde{g} \neq 0$, strongly $L^{q'}$. By weak convergence of $(g^m)_m$ in $L^q(Q)$ and strong convergence of $(\tilde{g}^m)_m$ in $L^{q'}(Q)$ we have that $\int_Q g\tilde{g} \geq \tilde{\delta} > 0$ and thus $g \not\equiv 0$.

The Lemma 2 says that, up to a subsequence, $(y^m)_m$ converges strongly to y in $L^q(Q)$ and by (3.5) the limit y is zero in Q_ω.

Now, if we invoque the Maximum Principle as recalled in the Preliminaries, since we have the hypothesis $L^0_{ij} \leq 0$ when $i \neq j$, then for an initial datum $y(0, \cdot) \geq 0$ and $g \geq 0$, we have that $y \geq 0$ in Q. Since y vanishes in Q_ω and thus in interior points of Q it implies that $y \equiv 0$ in Q which would imply that $g \equiv 0$ in Q, which is a contradiction.

Remark 4. *We used tha fact that L^q estimates in Theorem 3 remain valid with uniform constant for a whole family of such linear systems with fixed principal part and different first and zero order operators with coefficients in the corresponding L^1, L^0 satisfying an uniform bound in $L^\infty(Q)$. This fact is due to the use of Carleman estimates in which zero and first order terms may be absorbed in a uniform way if uniform bounds on the coefficients in lower order terms are available. Observe also that the L^q estimates remain valid for lower regularity of the coefficients of L^1, L^0: $b^k_i, c^l_i \in L^\infty(Q)$.*

4. L^∞ ESTIMATES FOR THE SOURCE TERM OF THE LINEAR SYSTEM.

The following remark, from [17], gives some consequences of the growth propery for the sources considered firstly in the paper of O.Yu. Imanuvilov and M.Yamamoto [13] ($|D_t g(t,x)| \leq \tilde{c}|g(\theta, x)|$) and says basically that the norm of the sources in Q is controlled by the norm of the sources in an arbitrary time neighborhood of the given moment θ:

Remark 5. *There exists a constant $C_1 = C_1(q, \tilde{c}) > 0$ such that for $g \in \mathcal{G}_{q,\tilde{\delta},\tilde{c},\tilde{G}}$ defined in* (1.7) *we have that*
$$\frac{1}{C_1}\|g(\theta,\cdot)\|_{L^q(\Omega)} \leq \|g\|_{L^q(Q)} \leq C_1 \|g(\theta,\cdot)\|_{L^q(\Omega)}. \tag{4.1}$$
Moreover, if $0 < \epsilon < \min(\theta, T-\theta)$, there exists $C_2 = C_2(\epsilon, q, \tilde{c}) > 0$ such that
$$\frac{1}{C_2}\|g\|_{L^q(Q^\epsilon)} \leq \|g\|_{L^q(Q)} \leq C_2 \|g\|_{L^q(Q^\epsilon)}. \tag{4.2}$$
Indeed, since $\left|\frac{\partial g}{\partial t}(t,x)\right| \leq \tilde{c}|g(t,x)|, a.e. (t,x) \in Q$, we have that
$$|g(t,x)| \leq \tilde{c}\int_\theta^t |g_t(\tau,x)| d\tau + |g(\theta,x)| \leq [|t-\theta|\tilde{c}+1]|g(\theta,x)|$$

$$\implies \|g\|^q_{L^q(Q)} \leq T(\tilde{c}T+1)^q \|g(\theta,\cdot)\|^q_{L^q(\Omega)}$$
$$\implies \|g\|_{L^q(Q)} \leq T^{\frac{1}{q}}(\tilde{c}T+1)\|g(\theta,\cdot)\|_{L^q(\Omega)},$$

giving the second inequality in (4.1).

Observe now that

$$|g(\theta,x)| \leq \tilde{c}\int_\theta^t |g_t(\tau,x)|d\tau + |g(t,x)| \leq \tilde{c}|t-\theta||g(\theta,x)| + |g(t,x)|$$

and if we consider $|t-\theta|<\delta<\frac{1}{2\tilde{c}}$, *we have that* $|g(\theta,x)| \leq 2|g(t,x)|$ *and thus integrating on* $(\theta-\delta,\theta+\delta)\times\Omega$ *we find*

$$2\delta\int_\Omega |g(\theta,x)|^q dx \leq 2^q \int_{|t-\theta|<\delta}\int_\Omega |g(\tau,x)|^q dx d\tau.$$

Using this, we find for $0<\delta<\min(\theta-\epsilon, T-(\theta-\epsilon))$

$$\|g(\theta,\cdot)\|_{L^q(\Omega)} \leq C\|g\|_{L^q(Q^\epsilon)},$$

where $C=C(\epsilon,\tilde{c},q)$. *This combined with the second inequality in* (4.1) *conclude both* (4.1) *and* (4.2).

We consider now the systems for the sources g and the corresponding systems obtained through derivation with respect to time:

$$\left\{\begin{array}{ll} D_t y + Ly + L^1 y + L^0 y = g, & \text{in } (0,T)\times\Omega, \\ \mathcal{B}y = 0, & \text{on } (0,T)\times\partial\Omega, \\ y(\theta,\cdot) = y_\theta, & \text{in } \Omega. \end{array}\right. \tag{4.3}$$

with $g \in \mathcal{G}_{q,\tilde{c},\tilde{\delta},\tilde{G}} \cap C^\alpha(Q)$, and for $z = D_t y$

$$\left\{\begin{array}{ll} D_t z + Lz + L^1 z + L^0 z = g_t - \tilde{L}y - \tilde{L}^1 y - \tilde{L}^0 y, & \text{in } (0,T)\times\Omega, \\ \mathcal{B}z = 0, & \text{on } (0,T)\times\partial\Omega, \\ z(\theta,\cdot) = g(\theta,\cdot) - Ly_\theta - L^1 y_\theta - L^0 y_\theta & \text{in } \Omega, \end{array}\right. \tag{4.4}$$

where

$$\tilde{L}y = (\tilde{L}_i y_i)^\top_{i=\overline{1,n}}, \quad \tilde{L}_i y_i = \sum_{j,l=1}^N D_j(D_t a_i^{jl} D_l y_i), i=\overline{1,n}$$
$$\tilde{L}^1 y = (\tilde{L}^1_i y)^\top_{i=\overline{1,n}}, \quad \tilde{L}^1_i y_i = \sum_{k=\overline{1,N}, l=\overline{1,n}} D_t b_i^{kl} D_k y_l, i=\overline{1,n} \tag{4.5}$$
$$\tilde{L}^0 y = (\tilde{L}^0_i y)^\top_{i=\overline{1,n}}, \quad \tilde{L}^0_i y_i = \sum_{l=1}^N D_t c_i^l y_l, i=\overline{1,n}.$$

In order to estimate L^∞ norm of g on $(0,T)\times\Omega$ in terms of the solution y in $(0,T)\times\omega$ and $y(\theta,\cdot)$ in Ω, we look at the third relation in (4.4) and for fixed $s_1>0$, we have that there exist constants denoted generically by C and depending on s_1, λ, $C = C(s_1, \lambda)$, such that

$$
\begin{aligned}
&\left\| g(\theta,\cdot)\, e^{s_1\alpha(\theta,\cdot)} \right\|_{L^\infty(\Omega)} \le \left\| g(\theta,\cdot)\, e^{s_1\alpha(\theta,\cdot)} \right\|_{C^\alpha(\Omega)} \\
&\le C\Big[\left\| z(\theta,\cdot)\, e^{s_1\alpha(\theta,\cdot)} \right\|_{C^\alpha(\Omega)} + \left\| L y_\theta e^{s_1\alpha} \right\|_{C^\alpha(\Omega)} \\
&+ \left\| L^1 y_\theta e^{s_1\alpha(\theta,\cdot)} \right\|_{C^\alpha(\Omega)} + \left\| L^0 y_\theta e^{s_1\alpha(\theta,\cdot)} \right\|_{C^\alpha(\Omega)} \Big] \\
&\le \left\| z(\theta,\cdot)\, e^{s_1\alpha(\theta,\cdot)} \right\|_{C^\alpha(\Omega)} + C(s_1,\lambda) \left\| y(\theta,\cdot) \right\|_{C^{2+\alpha}(\Omega)}.
\end{aligned}
\tag{4.6}
$$

We have to estimate the term $\left\| z(\theta,\cdot)\, e^{s_1\alpha(\theta,\cdot)} \right\|_{C^\alpha(\Omega)}$. We consider $q = \frac{N+1}{1-\alpha}$, in order to use the Morrey embedding theorem:

$$
\left\| z(\theta,\cdot)\, e^{s_1\alpha(\theta,\cdot)} \right\|_{C^\alpha(\Omega)} \le \left\| z e^{s_1\alpha} \right\|_{C^\alpha(Q)} \le C(\alpha) \left\| z e^{s_1\alpha} \right\|_{W^{1,q}(Q)}. \tag{4.7}
$$

Now, we estimate the last term above:

$$
\begin{aligned}
&\left\| z e^{s_1\alpha} \right\|_{W^{1,q}(Q)} = \\
&= \left\| z e^{s_1\alpha} \right\|_{L^q(Q)} + \left\| D_t (z e^{s_1\alpha}) \right\|_{L^q(Q)} + \left\| D(z e^{s_1\alpha}) \right\|_{L^q(Q)} \\
&\le C\left[s_1\lambda \left\| \varphi^2 z e^{s_1\alpha} \right\|_{L^q(Q)} + \left\| D_t z e^{s_1\alpha} \right\|_{L^q(Q)} + \left\| D z e^{s_1\alpha} \right\|_{L^q(Q)} \right].
\end{aligned}
\tag{4.8}
$$

At this point using (3) we obtain that for $s_2 = \frac{s_1}{\sigma}$, with $\sigma > 1$,

$$
s_1\lambda \left\| \varphi^2 z e^{s_1\alpha} \right\|_{L^q(Q)} \le C \left\| z e^{s_2\alpha} \right\|_{L^q(Q)} \tag{4.9}
$$

and going back to (4.8) we have,

$$
\left\| z e^{s_1\alpha} \right\|_{W^{1,q}(Q)} \le C\left[\left\| z e^{s_2\alpha} \right\|_{L^q(Q)} + \left\| D_t z e^{s_2\alpha} \right\|_{L^q(Q)} + \left\| D z e^{s_2\alpha} \right\|_{L^q(Q)} \right]. \tag{4.10}
$$

Using Carleman estimates with $s_3 = \frac{s_2}{\sigma}$, $\sigma > 1$, we obtain

$$
\begin{aligned}
&\| z e^{s_2\alpha} \|_{L^q(Q)} + \| D_t z e^{s_2\alpha} \|_{L^q(Q)} + \| D z e^{s_2\alpha} \|_{L^q(Q)} \\
&\le C[\| z e^{s_3\alpha} \|_{L^q(Q_\omega)} + \| g_t e^{s_3\alpha} \|_{L^q(Q)} \\
&+ \| \tilde{L} y e^{s_3\alpha} \|_{L^q(Q)} + \| \tilde{L}^1 y e^{s_3\alpha} \|_{L^q(Q)} + \| \tilde{L}^0 y e^{s_3\alpha} \|_{L^q(Q)}]
\end{aligned}
\tag{4.11}
$$

In the following we treat each term in the right side of the above inequality, keeping in mind that

$$
\| z e^{s_3\alpha} \|_{L^2(Q_\omega)} = \| D_t y e^{s_3\alpha} \|_{L^2(Q_\omega)} \le \| D_t y e^{s_3\alpha} \|_{L^q(Q)}
$$

We recall the estimates

$$
|\tilde{L} y| \le C(|D^2 y| + |Dy|), |\tilde{L}^1 y| \le C|Dy|, |\tilde{L}^0 y| \le C|y| \tag{4.12}
$$

for some constant C and $y \in W^{2,1}_q(Q)$. We may apply now appropriate Carleman estimates to system (4.3) with $s_4 = \frac{s_3}{\sigma}$, $\sigma > 1$, and obtain:

(4.13)
$$\|ze^{s_3\alpha}\|_{L^2(Q_\omega)} + \|\tilde{L}ye^{s_3\alpha}\|_{L^q(Q)} + \|\tilde{L}^1ye^{s_3\alpha}\|_{L^q(Q)} + \|\tilde{L}^0ye^{s_3\alpha}\|_{L^q(Q)}] \leq$$
$$\leq C(\|ge^{s_4\alpha}\|_{L^q(Q)} + \|ye^{s_4\alpha}\|_{L^2(Q_\omega)})$$

Going back to (4.8),
(4.14)
$$\|ze^{s_1\alpha}\|_{W^{1,q}(Q)} \leq C(\|ge^{s_4\alpha}\|_{L^q(Q)} + \|g_te^{s_3\alpha}\|_{L^q(Q)} + \|ye^{s_4\alpha}\|_{L^2(Q_\omega)}).$$

Now we use the fact that $g \in \mathcal{G}_{q,\tilde{\delta},\tilde{c},\tilde{G}}$ and this gives

$$\|ge^{s_4\alpha}\|_{L^q(Q)} + \|g_te^{s_3\alpha}\|_{L^q(Q)} \leq C\|g(\theta,\cdot)e^{s_4\alpha}\|_{L^q(Q)}$$

and (4.6) becomes
(4.15)
$$\left\|g\left(\theta,\cdot\right)e^{s_1\alpha(\theta,\cdot)}\right\|_{L^\infty(\Omega)}$$
$$\leq C(\|g(\theta,\cdot)e^{s_4\alpha}\|_{L^q(Q)} + \|ye^{s_4\alpha}\|_{L^2(Q_\omega)} + C(s_1,\lambda)\left\|y\left(\theta,\cdot\right)\right\|_{C^{2+\alpha}(\Omega)}).$$

Now, since $\theta = \frac{T}{2}$ and α attains its maximum in θ, we have that there exists $C = C(\Omega)$ such that

$$\|g(\theta,\cdot)e^{s_4\alpha}\|_{L^q(Q)} \leq \|g(\theta,\cdot)e^{s_4\alpha(\theta,\cdot)}\|_{L^q(Q)} \leq CT\|g(\theta,\cdot)\|_{L^q(\Omega)}. \tag{4.16}$$

Using the Remark 5 we get that there exists another constant C such that

$$\|g(\theta,\cdot)\|_{L^q(\Omega)} \leq C\|g\|_{L^q(Q)} \tag{4.17}$$

and using again Remark 5 and the L^q source estimate we have that

$$\begin{aligned} \|g\|_{L^\infty(Q)} &\leq C\left\|g\left(\theta,\cdot\right)e^{s_1\alpha(\theta,\cdot)}\right\|_{L^\infty(\Omega)} \\ &\leq C(\|ye^{s_4\alpha}\|_{L^q(Q_\omega)} + C(s_1,\lambda)\left\|y\left(\theta,\cdot\right)\right\|_{C^{2+\alpha}(\Omega)}). \end{aligned} \tag{4.18}$$

■

5. Source estimates for the reaction-diffusion system

We return to the system (1.4). In this section, the solutions of the nonlinear reaction-diffusion system are supposed to satisfy an apriori bound in $L^\infty(Q)$, and thus, for some $M > 0$, we assume $y \in \mathcal{F}_{q,M}$.

Proof of Theorem 1 under condition (A)

In order to apply the results from the linear case and taking into account the Remark 4, we consider the system that "approximates" the nonlinear reaction-diffusion system,

$$\begin{cases} D_ty + Ly + L^1y + L^0y = g, & (0,T)\times\Omega, \\ \mathcal{B}y = 0, & (0,T)\times\partial\Omega, \end{cases}, \tag{5.1}$$

with

$$D_ty = (D_ty_i)^\top_{i=\overline{1,n}}, Ly = (L_iy_i)^\top_{i=\overline{1,n}}, L^1y = \left(L^1_iy_i\right)^\top_{i=\overline{1,n}}$$

and the coupling is done through

$$L^0 y = \left(L_i^0 y\right)_{i=\overline{1,n}}^{\top},\ L_i^0 y = \sum_{j=1}^{n} \gamma_i^j y_j$$

where

$$\gamma_i^j(t,x) := \int_0^1 \frac{\partial f_i(\tau y_1, ..., \tau y_j, ...\tau y_n)}{\partial y_j} d\tau. \tag{5.2}$$

Since $f_i, i = \overline{1,n}$ are $C^\infty(\mathbb{R}^n)$ functions, $f(0) = 0$ and $y = (y_i)_{i=\overline{1,n}}^{\top} \in \mathcal{F}_{q,M}$, we have that there exists a constant $m_0 > 0$ such that

$$|\gamma_i^j| \leq m_0, \forall i, j = \overline{1,n}.$$

Moreover, the above conditions and the hypothesis (H1) on the nonlinearities allow to apply the results from the linear case. In this context, if we consider sources g form $\mathcal{G}_{q,\tilde{\delta},\tilde{G}}$, for some $\tilde{\delta} > 0$ we can apply the result from Theorem 3 concerning the L^q estimates to the above linear system to obtain the L^q estimates for the source of the reaction-diffusion system.

Proof of Theorem 1 under condition (B).

Consider now the reaction diffusion system under sources g from $\mathcal{G}_{q,\tilde{\delta},\tilde{G}}$ with the hypothesis from condition (B) satisfied. We want to prove that there exists a constant $C > 0$ such that

$$\|g\|_{L^q(Q)} \leq C\|y\|_{L^q(Q_\omega)}.$$

We reason by contradiction. Suppose that there exists a sequence $(g^m)_m \in \mathcal{G}_{q,\tilde{\delta},\tilde{G}}$ and the corresponding solutions $(y^m)_m \in \mathcal{F}_{q,M}$ such that

$$\|g^m\|_{L^q(Q)} > m\|y^m\|_{L^q(Q_\omega)}, \forall m > 0. \tag{5.3}$$

We will treat the cases when $\|g^m\|_{L^q(Q)} \longrightarrow \infty$, $\|g^m\|_{L^q(Q)} \longrightarrow 0$ and when $\|g^m\|_{L^q(Q)}$ bounded but $\|g^m\|_{L^q(Q)} \not\longrightarrow 0$.

Case 1. If $\|g^m\|_{L^q(Q)} \longrightarrow \infty$, we consider the linear system by the sequence $(w^m)_m$, $w^m = \frac{y^m}{\|g^m\|_{L^q(Q)}}$:

$$\begin{aligned} &D_t w^m + L w^m + L^1 w^m = h^m - \frac{1}{\|g^m\|_{L^q(Q)}} f(y^m), (0,T) \times \Omega \\ &\mathcal{B} w^m = 0, (0,T) \times \partial\Omega, \end{aligned} \tag{5.4}$$

where $h^m := \frac{1}{\|g^m\|_{L^q(Q)}} g^m$.

By Lemma 2 applied to the pair (w^m, h^m) we have that there exist $h \in L^q(Q)$ and $w \in L^q_{loc}(0,T; W^{1,q}(\Omega))$ such that, up to subsequences denoted in the same manner, we have

$$h^m \rightharpoonup h \text{ weakly in } L^q(Q) \tag{5.5}$$

and

$$w^m \longrightarrow w \text{ strongly in } L^q_{loc}(0,T;W^{1,q}(\Omega)) \tag{5.6}$$

and h, w verify the problem

$$\begin{aligned} &D_t w + Lw + L^1 w = h, (0,T)\times\Omega \\ &\mathcal{B}w = 0, (0,T)\times\partial\Omega. \end{aligned} \tag{5.7}$$

By the hypothesis 5.3, we have that the limit w satisfies

$$w = 0 \text{ in } Q_\omega.$$

Moreover, observe that $h \not\equiv 0$ in Q. Indeed, by hypotheses $g^m \in \mathcal{G}_{q,\tilde{\delta},\tilde{G}}$ we have that $h^m \in \mathcal{G}_{q,\tilde{\delta},\tilde{G}}$ and thus there exist a corresponding $\tilde{g}^m \in \tilde{G}$ such that

$$\int_Q h^m \tilde{g}^m \geq \tilde{\delta}\|h^m\|_{L^q(Q)} = \tilde{\delta}.$$

Extracting a subsequence we may suppose that $\tilde{g}_m \to \tilde{g} \in \tilde{G}, \tilde{g} \neq 0$, strongly $L^{q'}$. By weak convergence of $(h^m)_m$ in $L^q(Q)$ and strong convergence of $(\tilde{g}^m)_m$ in $L^{q'}(Q)$ we have that $\int_Q h\tilde{g} \geq \tilde{\delta} > 0$ and thus $h \not\equiv 0$. Then, by the Maximum principle applied to the system (5.7), we have that $w \geq 0$ in Q. Since $w = 0$ in Q_ω, it follows that $w \equiv 0$ in Q, which implies that $h \equiv 0$ in Q, which is false.

Case 2. If $\|g^m\|_{L^q(Q)}$ bounded but $\|g^m\|_{L^q(Q)} \longrightarrow \mu \neq 0$ we have that up to a subsequence, $g^m \rightharpoonup g$ weakly in $L^q(Q)$ and $g \not\equiv 0$ in Q.

The sequences $(g^m)_m$ and $(y^m)_m$ satisfy the problems

$$\begin{cases} D_t y^m + Ly^m + L^1 y^m = g^m - f(y^m), & \text{in } (0,T)\times\Omega, \\ \mathcal{B}y^m = 0, & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{5.8}$$

Since $(y^m)_m \subset \mathcal{F}_{q,M}$, we have that $(y^m)_m$ is bounded in $L^\infty(Q)$ and, since f is smooth, we have for some $K > 0$ a bound in L^∞ for the term $\|f(y^m)\|L^\infty(Q) \leq K, m \in \mathbb{N}$. By Carleman estimates we have that for $0 < \epsilon < \min(\theta, T-\theta)$, there exists a constant $C = C(\epsilon) > 0$ such that

$$\|y^m\|_{W^{2,1}_q(Q^\epsilon)} \leq C(\|g^m\|_{L^q(Q)} + \|y^m\|_{L^q(Q_\omega)} + K),$$

where Q^ϵ is the cylinder $Q^\epsilon = (\epsilon, T-\epsilon)\times\Omega$. By Aubin-Lions lemma applied to sequence $(y^m)_m$, there exists $y \in L^q_{loc}(0,T;W^{1,q}(\Omega))$ such that up to subsequences of y^m

$$y^m \longrightarrow y \text{ in } L^q(\epsilon, T-\epsilon; W^{1,q}(\Omega)) \text{ as } m \longrightarrow \infty, \forall \epsilon > 0. \tag{5.9}$$

Strong convergence of $(y^m)_m$ in $L^q(\epsilon, T-\epsilon; W^{1,q}(\Omega))$ and weak convergence of $(g^m)_m$ in $L^q(Q)$ allow to pass to the limit in the variational formulation of problem to obtain that y is the solution of the nonlinear system with source g:

$$\begin{aligned} &D_t y + Ly + L^1 y = g - f(y), (0,T)\times\Omega \\ &\mathcal{B}y = 0, (0,T)\times\partial\Omega. \end{aligned} \tag{5.10}$$

Moreover, by (5.3), $y \equiv 0$ in $\omega \times (0,T)$. By hypothesis (H_2), invoking the invariance principle from [16] we get $y \geq 0$ in $(\epsilon, T-\epsilon) \times \Omega$ for all $\epsilon > 0$, and consequently in whole Q.

Now, since $q > \frac{N+2}{2}$, the Sobolev embedding gives that $W_q^{2,1}(Q) \subset C^{\alpha}(Q)$ and so y is Hölder continuous in Q for some $\alpha \in (0,1)$. This implies that y is uniformly continuous in $\overline{Q}$ and so there exists δ_0 independent of ω such that in each neighborhood $(0,T) \times [(\omega + B_{\delta_0}) \cap \Omega]$ we have that y remains in the domain of quasimonotonicity of f. In this slightly larger domain we consider again the "approximate" linear system (5.1)

$$\begin{cases} D_t y + Ly + L^1 y + L^0 y = g, & (0,T) \times \Omega, \\ \mathcal{B}y = 0, & (0,T) \times \partial\Omega, \end{cases}, \tag{5.11}$$

and we use the same method (based on the maximum principle for linear coupled systems, [18]) as when working under the condition (A) to obtain that $y \equiv 0$ in $(0,T) \times [(\omega + B_{\delta_0}) \cap \Omega]$; consequently we may extend the vanishing property to the whole cylinder $(0,T) \times \Omega$. This would imply that $g \equiv 0$ in Q, which is a contradiction.

Case 3. When $\|g^m\|_{L^q(Q)} \to 0$, we consider the linear system

$$\begin{aligned} & D_t y^m + Ly^m + L^1 y^m + L^0_m y^m = g^m, (0,T) \times \Omega \\ & \mathcal{B}y^m = 0, (0,T) \times \partial\Omega, \end{aligned} \tag{5.12}$$

where

$$L^0_m y^m = \left(L^0_{i,m} y^m\right)^{\top}_{i=\overline{1,n}}, \; L^0_{i,m} y^m = \sum_{j=1}^{n} \gamma_i^{j,m} y_j^m$$

with

$$\gamma_i^{j,m}(t,x) := \int_0^1 \frac{\partial f_i(\tau y_1^m, ..., \tau y_j^m, ... \tau y_n^m)}{\partial y_j^m} d\tau. \tag{5.13}$$

Along with (5.12) we consider the system verified by $w^m := \frac{1}{\|g^m\|_{L^q(Q)}} y^m$

$$\begin{aligned} & D_t w^m + Lw^m + L^1 w^m + L^0_m w^m = h^m, (0,T) \times \Omega \\ & \mathcal{B}w^m = 0, (0,T) \times \partial\Omega, \end{aligned} \tag{5.14}$$

where $h^m := \frac{1}{\|g^m\|_{L^q(Q)}} g^m$. One may observe that $\|h^m\|_{L^q(Q)} = 1$ and so there exists a subsequence denoted also $(h^m)_m$ and $h \in L^q(Q)$ such that $h^m \rightharpoonup h$ weakly in $L^q(Q)$. Looking at the properties of the set of sources, one may see that $h \not\equiv 0$ in Q.

We would like to pass to limit in the above system. Since $(y^m)_m$ is in $\mathcal{F}_{q,M}$, we have that the entries of L^0_m are bounded in $L^\infty(Q)$, and up to a subsequence,

$$\gamma_i^{j,m} \overset{w^*}{\rightharpoonup} \gamma_i^j \text{ weakly}^* \text{ in } L^\infty.$$

Using the hypothesis (5.3) and the global Carleman estimate for the system (5.12) we get that $(w^m)_m$ is bounded in $W_q^{2,1}(Q)$. By Aubin-Lions lemma, we obtain that there exists $w \in L^q_{loc}(0,T;W^{1,q}(\Omega))$ such that, up to a subsequence, we have

$$w^m \longrightarrow w \text{ in } L^q_{loc}(0,T;W^{1,q}(\Omega)). \tag{5.15}$$

At this point we can pass to the limit in the variational formulation of (5.12) and find that w verifies the equation with source h in Q:

$$\begin{aligned} &D_t w + Lw + L^1 w + L^0 w = h, (0,T)\times\Omega \\ &\mathcal{B}w = 0, (0,T)\times\partial\Omega, \end{aligned} \tag{5.16}$$

where the coefficients γ_i^j of L^0 are the weak* limits in $L^\infty(Q)$ of the sequences of corresponding coefficients of L^0_m. Moreover, again by hypothesis 5.3, we have that the limit w satisfies

$$w = 0 \text{ in } Q_\omega.$$

We work in hypothesis **(B)**, so $q > \frac{N+2}{2}$ and the Sobolev embedding $W_q^{2,1}(Q) \subset C^\alpha(Q)$ for some $\alpha \in (0,1)$ together with a Carleman estimate for the solution y^m of the (5.12) gives that

$$\|y^m\|_{L^\infty(Q^\epsilon)} \longrightarrow 0, \forall \epsilon > 0.$$

This observation alows to place y^m in the ε_0-neighbourhood, $\mathcal{V}_{\varepsilon_0}(0)$ where $(H1)$ holds, when $(t,x) \in Q^\epsilon$. Consequently, $\gamma_i^{j,m} \le 0$ in Q^ϵ for m big enough and passing to the limit we find $\gamma_i^j \le 0$ in Q. We may now apply the maximum principle for the linear parabolic system, find that $w \equiv 0$ in Q and thus $h \equiv 0$ in Q which is a contradiction. ■

Proof of Theorem 2

Concerning L^∞ source estimates, we consider the reaction-diffusion system with a given observation y_θ at the instant $\theta \in (0,T)$,

$$\begin{cases} D_t y + Ly + L^1 y + f(y) = g, & (0,T)\times\Omega, \\ \mathcal{B}y = 0, & (0,T)\times\partial\Omega, \\ y(\theta,\cdot) = y_\theta & \text{in } \Omega. \end{cases} \tag{5.17}$$

From the above system, using the mean value theorem, we obtain the linear system that "approximates" the nonlinear reaction-diffusion system, (5.1):

$$\begin{cases} D_t y + Ly + L^1 y + L^0 y = g, & (0,T)\times\Omega, \\ \mathcal{B}y = 0, & (0,T)\times\partial\Omega, \ , \\ y(\theta,\cdot) = y_\theta & \text{in } \Omega, \end{cases} \tag{5.18}$$

where $L^0 y = (L^0_i y)^\top, L^0_i y = \sum_{j=1}^n \gamma_i^j y_j$, γ_i^j defined in (5.2).

From (5.17) we obtain through derivation with respect to time a linear system for the variable $z = D_t y$:

(5.19)
$$\begin{cases} D_t z + Lz + L^1 z + Df(y) z = D_t g - \tilde{L} y - \tilde{L}^1 y, & (0,T)\times\Omega, \\ \mathcal{B} z = 0, & (0,T)\times\partial\Omega, \\ z(\theta,\cdot) = g(\theta,\cdot) - L y_\theta - L^1 y_\theta - L^0 y_\theta & \text{in } \Omega, \end{cases}$$

where $\tilde{L}, \tilde{L}^1$ are given in (4.5).

Remark 6. *We want to apply the same procedure as in the case of linear systems in §4 , based on combining L^q-Carleman estimates for y and $z = D_t y$. There we used appropriate Carleman estimates for the linear systems satisfied by z and y, which are in this section represented by* (5.18) *and* (5.19). *In the present case we have in fact families of such linear systems and the Carleman estimates hold with uniform constants if L^∞ bounds on the coefficients of zero and first order terms are assumed. In fact we have families of linear systems for y, z with variable zero order terms, with coefficients γ_i^j and, respectively, $Df(y)$, which are uniformly bounded in $L^\infty(Q)$ as a consequence of the assumption $y \in \mathcal{F}_{q,M}$.*

We start with the linear system (5.19) and we treat it like we did with (4.4). From the last equation in (5.19), for $s_1 > 0$, we obtain that there exists $C(s_1,\lambda) > 0$ such that

$$(5.20) \qquad \begin{aligned} &\left\| g(\theta,\cdot)\, e^{s_1\alpha(\theta,\cdot)} \right\|_{L^\infty(\Omega)} \le \left\| g(\theta,\cdot)\, e^{s_1\alpha(\theta,\cdot)} \right\|_{C^\alpha(\Omega)} \\ &\le \left\| z(\theta,\cdot)\, e^{s_1\alpha(\theta,\cdot)} \right\|_{C^\alpha(\Omega)} + C(s_1,\lambda) \left\| y(\theta,\cdot) \right\|_{C^{2+\alpha}(\Omega)} . \end{aligned}$$

To estimate the term $\left\| z(\theta,\cdot)\, e^{s_1\alpha(\theta,\cdot)} \right\|_{C^\alpha(\Omega)}$ we use the Morrey embedding theorem:

$$(5.21) \quad \left\| z(\theta,\cdot)\, e^{s_1\alpha(\theta,\cdot)} \right\|_{C^\alpha(\Omega)} \le \left\| z e^{s_1\alpha} \right\|_{C^\alpha(Q)} \le C(\alpha) \left\| z e^{s_1\alpha} \right\|_{W^{1,q}(Q)} .$$

For the term $\| z e^{s_1\alpha} \|_{W^{1,q}(Q)}$ we apply a Carleman inequality to system (5.19),

$$(5.22) \qquad \begin{aligned} &\| z e^{s_2\alpha} \|_{L^q(Q)} + \| D_t z e^{s_2\alpha} \|_{L^q(Q)} + \| D z e^{s_2\alpha} \|_{L^q(Q)} \\ &\le C[\| z e^{s_3\alpha} \|_{L^q(Q_\omega)} + \| g_t e^{s_3\alpha} \|_{L^q(Q)} \\ &+ \| \tilde{L} y e^{s_3\alpha} \|_{L^q(Q)} + \| \tilde{L}^1 y e^{s_3\alpha} \|_{L^q(Q)} + \| \tilde{L}^0 y e^{s_3\alpha} \|_{L^q(Q)}], \end{aligned}$$

where $s_2 = s_1/\sigma, s_3 = s_2/\sigma$ for fixed $\sigma > 1$ and the constant in the above estimate $C = C(s_1,\lambda,\sigma)$.

At this point we couple the two systems (5.18) and (5.19), by using a Carleman estimate for (5.18), to bound the weighted terms in y:

(5.23)

$$\|D_t y e^{s_3\alpha}\|_{L^2(Q_\omega)} + \|\tilde{L} y e^{s_3\alpha}\|_{L^q(Q)} + \|\tilde{L}^1 y e^{s_3\alpha}\|_{L^q(Q)} + \|\tilde{L}^0 y e^{s_3\alpha}\|_{L^q(Q)} \leq$$
$$\leq C(\|g e^{s_4\alpha}\|_{L^q(Q)} + \|y e^{s_4\alpha}\|_{L^q(Q_\omega)}),$$

where $C = C(s_1, \lambda, \sigma)$, $s_4 = s_3/\sigma$. We use that $g \in \mathcal{G}_{q,\tilde{\delta},\tilde{c},\tilde{G}}$ to be able to write

$$\|g e^{s_4\alpha}\|_{L^q(Q)} + \|g_t e^{s_3\alpha}\|_{L^q(Q)} \leq C\|g(\theta,\cdot)e^{s_4\alpha}\|_{L^q(Q)}$$

and gathering the estimates above, using Remark 5 and the fact that α attains its maximum in $\theta = \frac{T}{2}$, we obtain

$$\begin{aligned} &\|g\|_{L^\infty(Q)} \leq C(s_1, \lambda, \tilde{c}) \left\|g\left(\theta,\cdot\right) e^{s_1\alpha(\theta,\cdot)}\right\|_{L^\infty(\Omega)} \\ &\leq C(\|g\|_{L^q(Q)} + \|y e^{s_4\alpha}\|_{L^q(Q_\omega)} + C(s_1,\lambda)\left\|y\left(\theta,\cdot\right)\right\|_{C^{2+\alpha}(\Omega)}). \end{aligned} \tag{5.24}$$

At this point we use the L^q source estimates obtained for g under conditions (A) or (B) to properly bound the term $\|g\|_{L^q(Q)}$ from the right-hand side and to get the desired L^∞ source estimate (1.12), for the nonlinear reaction diffusion system (1.4). ∎

References

[1] Herbert Amann. Invariant sets and existence theorems for semilinear parabolic and elliptic systems. *J. Math. Anal. Appl.*, 65(2):432–467, 1978.

[2] Viorel Barbu. Controllability of parabolic and Navier-Stokes equations. *Sci. Math. Jpn.*, 56(1):143–211, 2002.

[3] Viorel Barbu. The Carleman inequality for linear parabolic equations in L^q norm. *Differential Integral Equations*, 15(5):513–525, 2002.

[4] Mourad Choulli. *Une introduction aux problèmes inverses elliptiques et paraboliques*, volume 65 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Berlin, 2009.

[5] Jean-Michel Coron, Sergio Guerrero, and Lionel Rosier. Null controllability of a parabolic system with a cubic coupling term. *SIAM J. Control Optim.*, 48(8):5629–5653, 2010.

[6] Jean-Michel Coron and Jean-Philippe Guilleron. Control of three heat equations coupled with two cubic nonlinearities. *SIAM J. Control Optim.*, 55(2):989–1019, 2017.

[7] L. Desvillettes, Th. Lepoutre, and A. Moussa. Entropy, duality, and cross diffusion. *SIAM J. Math. Anal.*, 46(1):820–853, 2014.

[8] Laurent Desvillettes and Klemens Fellner. Exponential decay toward equilibrium via entropy methods for reaction-diffusion equations. *J. Math. Anal. Appl.*, 319(1):157–176, 2006.

[9] Laurent Desvillettes and Klemens Fellner. Entropy methods for reaction-diffusion equations: slowly growing a-priori bounds. *Rev. Mat. Iberoam.*, 24(2):407–431, 2008.

[10] Laurent Desvillettes, Klemens Fellner, and Bao Quoc Tang. Trend to equilibrium for reaction-diffusion systems arising from complex balanced chemical reaction networks. *SIAM J. Math. Anal.*, 49(4):2666–2709, 2017.

[11] Lawrence Christopher Evans. A strong maximum principle for parabolic systems in a convex set with arbitrary boundary. *Proc. Amer. Math. Soc.*, 138(9):3179–3185, 2010.

[12] A.V. Fursikov and O.Yu. Imanuvilov. *Controllability of evolution equations.* Seoul: Seoul National Univ., 1996.

[13] Oleg Yu Imanuvilov and Masahiro Yamamoto. Lipschitz stability in inverse parabolic problems by the Carleman estimate. *Inverse Probl.*, 14(5):1229–1245, 1998.

[14] O. A. Ladyzenskaja, V. A. Solonnikov, and N. N. Uralceva. *Linear and quasi-linear equations of parabolic type.* Vol. 23. American Mathematical Society, Providence, R.I., 1968.

[15] Kévin Le Balc'h. Controllability of a 4×4 quadratic reaction-diffusion system. *J. Differential Equations*, 266(6):3100–3188, 2019.

[16] Robert H. Martin, Jr. A maximum principle for semilinear parabolic systems. *Proc. Amer. Math. Soc.*, 74(1):66–70, 1979.

[17] Elena-Alexandra Melnig. Stability in L^q-norm for inverse source parabolic problems. *Submitted to Journal of Inverse and Ill-posed Problems.*

[18] Murray H. Protter and Hans F. Weinberger. *Maximum principles in differential equations.* Springer-Verlag, New York, 1984. Corrected reprint of the 1967 original.

[19] Hans F. Weinberger. Invariant sets for weakly coupled parabolic and elliptic systems. *Rend. Mat. (6)*, 8:295–310, 1975. Collection of articles dedicated to Mauro Picone on the occasion of his ninetieth birthday.