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Data dependence multidifferentiability and systems in variations: a counterexample

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Abstract

The paper concerns ordinary differential equations under standard Carathéodory - Lipschitz assumptions. In this setting, data dependence is always multidifferentiable, and moreover, the multidifferential has all the features of the Fréchet differential except for the quality of being an additive function. In addition, the multidifferential always comes under a convenient system in variations, but the converse may fail outside the function setting. In this regard, a counterexample is given.

1 Introduction

Consider the differential system

$$\begin{cases} \dot{y}(t) = f(t, y(t)), \\ y(a) = x, \end{cases} \tag{1}$$

where $[a, b] \subseteq R$ is a closed interval, $\Omega \subseteq R^n$ is an open set, and the function $f : [a, b] \times \Omega \to R^n$ satisfies some Carathéodory conditions of measurability, continuity, and boundedness:

- for every $x \in \Omega$ the function $f(\cdot, x)$ is measurable;
- for almost every $t \in [a, b]$ the function $f(t, \cdot)$ is continuous;

- there exists an open covering $\mathcal{O}$ of Ω such that for every $O \in \mathcal{O}$ there exists an integrable function $m : [a, b] \to R$ such that for almost every $t \in [a, b]$ and for every $x \in O$ there holds the inequality

$$\|f(t, x)\| \leq m(t).$$

Then for every initial datum $x \in \Omega$ there exists at least a subinterval $[a, c] \subseteq [a, b]$ and a Carathéodory solution $y : [a, c] \to R^n$ to the differential system (1). Further, assume f satisfies a Lipschitz condition:

- there exists an open covering $\mathcal{O}$ of Ω such that for every $O \in \mathcal{O}$ there exists an integrable function $m : [a, b] \to R$ such that for almost every $t \in [a, b]$, for every $x_1 \in O$, and for every $x_2 \in O$ there holds the inequality
$$\|f(t, x_2) - f(t, x_1)\| \leq m(t)\|x_2 - x_1\|.$$

Then for every initial datum $x \in \Omega$ and for every subinterval $[a, c] \subseteq [a, b]$ there exists at most a Carathéodory solution $y : [a, c] \to R^n$ to the differential system (1).

Finally, assume there exists at least an initial datum $x \in \Omega$ such that the differential system (1) has a Carathéodory solution $y : [a, b] \to R^n$, denote by Σ the set of these initial data, and for each $x \in \Sigma$ denote the corresponding $y : [a, b] \to R^n$ by $\sigma(x)$. The set Σ is open and the function $\sigma : \Sigma \to \mathcal{C}([a, b]; R^n)$ is lipschitzean:

- there exists an open covering $\mathcal{O}$ of Ω such that for every $O \in \mathcal{O}$ there exists a real munber $\lambda > 0$ such that for every $x_1 \in O$ and for every $x_2 \in O$ there holds the inequality

$$\|\sigma(x_2) - \sigma(x_1)\| \leq \lambda\|x_2 - x_1\|.$$

In Sections 2, there are recalled the results which state that the function σ is multidifferentiable at every point, and moreover, the multidifferentials have all the features of the Fréchet differential except for point-valued-ness and additive-ness. In Section 3, it is pointed out that almost all functions $f(t, \cdot)$ have the same multidifferentiability properties. In addition, it is proved that the multidifferential of σ comes under a system in variations based upon the multidifferentials of $f(t, \cdot)$. The converse holds too if the multidifferentials of $f(t, \cdot)$ are point-valued, otherwise the converse may fail. A counterexample in this regard is given in Section 4.

2 Data dependence multidifferentiability

Let $x \in \Sigma$ and define the multifunction $K_\sigma(x) : R^n \to \mathcal{C}([a,b]; R^n)$ through the equality of Aubin [1, p. 235]

$$\mathrm{graph}(K_\sigma(x)) = K_{\mathrm{graph}(\sigma)}(x, \sigma(x))$$

where K stands for the tangency concept of Bouligand [3, p. 32] and Severi [4, p. 99]. This means that $v \in K_\sigma(x)(u)$ if and only if

- for every neighborhood Q of the origin in $\mathcal{C}([a,b]; R^n)$, for every neighborhood P of the origin in R^n, and for every real number $r > 0$ there exist $s \in (0,r)$ and $p \in P$ such that there holds the relation

$$(1/s)(\sigma(x + s(u+p)) - \sigma(x)) \in v + Q.$$

Data dependence multidifferentiability essentialy depends on a relative compactness result.

Theorem 1 *There exists a neighborhood P of the origin in R^n such that it is relatively compact the $\mathcal{C}([a,b]; R^n)$-set*

$$\{(1/s)(\sigma(x + sp) - \sigma(x)); s \in (0,1], p \in P\}.$$

Proof. First, there exist real numbers $\varpi > 0$ and $\lambda > 0$ such that $x + p \in \Sigma$ and $\|\sigma(x+p) - \sigma(x)\| \le \lambda\|p\|$ for all $p \in R^n$ with $\|p\| \le \varpi$. Since $\sigma(x)([a,b])$ is a compact subset of Ω, we can suppose (taking a smaller ϖ if necessary) that there exists an integrable function $m : [a,b] \to R$ such that $\xi + q \in \Omega$ and $\|f(t, \xi + q) - f(t, \xi)\| \le m(t)\|q\|$ for almost all $t \in [a,b]$, for all $\xi \in \sigma([a,b])$, and for all $q \in R^n$ with $\|q\| \le \lambda\varpi$.

Now, let $s \in (0,1]$, let $p \in R^n$ with $\|p\| \le \varpi$, note $x + sp \in \Sigma$, and set $w = (1/s)(\sigma(x+sp) - \sigma(x))$. Then $\|w(t)\| \le \lambda\|p\| \le \lambda\varpi$ for all $t \in [a,b]$, therefore

$$\|\dot{w}(t)\| = \|(1/s)(f(t, \sigma(x)(t) + sw(t)) - f(t, \sigma(x)(t)))\| \le m(t)\lambda\varpi$$

for almost all $t \in [a,b]$, and the conclusion follows. *QED*

According to a result in [8, p. 248, Theorem 5.1], the positively homogeneous multifunction $K_\sigma(x)$ has all the features of the Severi [5, p. 10, eq. (6)] directional differential except for the quality of being a function:

- for every $u \in R^n$ the subset $K_\sigma(x)(u)$ of $\mathcal{C}([a,b]; R^n)$ is nonempty and compact; in particular, $K_\sigma(x)(0) = \{0\}$;

- for every $u \in R^n$ and for every neighborhood Q of the origin in $\mathcal{C}([a,b];R^n)$ there exists a neighborhood P of the origin in R^n and a real number $r > 0$ such that for every $s \in (0,r)$ and for every $p \in P$ there holds the relation

$$(1/s)(\sigma(x + s(u + p)) - \sigma(x)) \in K_\sigma(x)(u) + Q.$$

This **multidifferentiability** of σ at x does hold in case of no matter which function σ which is defined on an open subset Σ of a Hausdorff vector space X, which takes values in a Hausdorff vector space Y, and which satisfies the above property of relative compact-ness at x.

Accordingly the **multidifferential** $K_\sigma(x)$ of σ at x is upper semicontinuous, but if X as well as Y are normed spaces and it is finite the Lipschitz number

$$\Lambda_\sigma(x) = \limsup_{\substack{x_1 \to x \\ x_2 \to x \\ x_2 \neq x_1}} \frac{\|\sigma(x_2) - \sigma(x_1)\|}{\|x_2 - x_1\|},$$

then the arguments in Blagodatskih [2, p. 2138, 3°, 3)] prove that the multidifferential $K_\sigma(x)$ is not only upper semicontinuous, but also satifies the Lipschitz inequality

$$\sup_{v_2 \in K_\sigma(x)(u_2)} \inf_{v_1 \in K_\sigma(x)(u_1)} \|v_2 - v_1\| \leq \Lambda_\sigma(x)\|u_2 - u_1\|$$

for all $u_1 \in X$ and for all $u_2 \in X$. In addition, if the space X is finite dimensional and the multifunction $K_\sigma(x)$ is a function, then the **differential** $K_\sigma(x)$ has all the features of the Fréchet differential except for additive-ness:

$$\lim_{\substack{u \to 0 \\ u \neq 0}} \|\sigma(x + u) - \sigma(x) - K_\sigma(x)(u)\|/\|u\| = 0$$

(see [6, p. 202, Proposition 8]).

3 Systems in variations

Let $x \in \Sigma$, let $y = \sigma(x)$, and consider the system in variations

$$\begin{cases} \dot{v}(t) \in \mathrm{co}\{K_{f(t,\cdot)}(y(t))(v(t))\}, \\ v(a) = u \end{cases} \tag{2}$$

(cf Blagodatskih [2, p. 2138, Theorem 2]). Here co stands for *convex hull*, whereas for every $t \in [a,b]$ and for every $\xi \in R^n$ the multifunction $K_{f(t,\cdot)}(\xi) : R^n \to R^n$ is defined through the equality

$$\mathrm{graph}(K_{f(t,\cdot)}(\xi)) = K_{\mathrm{graph}(f(t,\cdot))}(\xi, f(t,\xi)).$$

This means that $\psi \in K_{f(t,\cdot)}(\xi)(\chi)$ if and only if

- for every neighborhood Q of the origin in R^n, for every neighborhood P of the origin in R^n, and for every real number $r > 0$ there exist $s \in (0, r)$ and $p \in P$ such that there holds the relation

$$(1/s)(f(t, \xi + s(\chi + p)) - f(t, \xi)) \in \psi + Q.$$

Since no matter which open covering in R^n has a countable subcover, it follows that for almost every $t \in [a, b]$ and for every $\xi \in \Omega$ it is finite the Lipschitz number

$$\Lambda_{f(t,\cdot)}(\xi) = \limsup_{\substack{\xi_1 \to 0 \\ \xi_2 \to 0 \\ \xi_2 \neq \xi_1}} \frac{\|f(t, \xi_2) - f(t, \xi_1)\|}{\|\xi_2 - \xi_1\|},$$

therefore there exists a neighborhood P of the origin in R^n such that it is bounded, hence relatively compact the R^n-subset

$$\{(1/s)(f(t, \xi + sp) - f(t, \xi)); s \in (0, 1], p \in P\},$$

so the function $f(t, \cdot)$ is **multidifferentiable** at ξ, i.e. the positively homogeneous multifunction $K_{f(t,\cdot)}(\xi)$ has all the features of the Severi directional differential except for the property of being a function:

- for every $\chi \in R^n$ the subset $K_{f(t,\cdot)}(\xi)(\chi)$ of R^n is nonempty and compact; in particular, $K_{f(t,\cdot)}(\xi)(0) = \{0\}$;

- for every $\chi \in R^n$ and for every neighborhood Q of the origin in R^n there exist a neighborhood P of the origin in R^n and a real number $r > 0$ such that for every $s \in (0, r)$ and for every $p \in P$ there holds the relation

$$(1/s)(f(t, \xi + s(\chi + p)) - f(t, \xi)) \in K_{f(t,\cdot)}(\xi)(\chi) + Q.$$

Accordingly the **multidifferential** $K_{f(t,\cdot)}(\xi)$ satisfies a Lipschitz inequality

$$\sup_{\psi_2 \in K_{f(t,\cdot)}(\xi)(\chi_2)} \inf_{\psi_1 \in K_{f(t,\cdot)}(\xi)(\chi_1)} \|\psi_2 - \psi_1\| \leq \Lambda_{f(t,\cdot)}(\xi)\|\chi_2 - \chi_1\|$$

for all $\chi_1 \in R^n$ and for all $\chi_2 \in R^n$. In addition, if the multifunction $K_{f(t,\cdot)}(\xi)$ is a function, then the **differential** $K_{f(t,\cdot)}(\xi)$ has all the features of the Fréchet differential except for additive-ness:

$$\lim_{\substack{\chi \to 0 \\ \chi \neq 0}} \|f(t, \xi + \chi) - f(t, \xi) - K_{f(t,\cdot)}(\xi)(\chi)\|/\|\chi\| = 0.$$

Note parenthetically that, in view of the differentiability properties above, the core of the Rademacher - Stepanov result states just that, except for a set of points $\xi \in \Omega$ which has a null Lebesgue measure, the positively homogeneous multifunction $K_{f(t,\cdot)}(\xi)$ is an additive function.

Theorem 2 *Let $u \in R^n$ and let $v \in K_\sigma(x)(u)$. Then v is a Carathéodory solution to the system in variations (2).*

Proof. Consider a sequence $(s_i, u_i, v_i) \in (0, +\infty) \times R^n \times \mathcal{C}([a, b]; R^n)$ which converges to $(0, u, v)$ and which satisfies the relation $(1/s_i)(\sigma(x+s_i u_i) - \sigma(x)) = v_i$. Then $v_i : [a, b] \to R^n$ is a Carathéodory solution to the differential system

$$\begin{cases} \dot{v}_i(t) = (1/s)(f(t, y(t) + s_i v_i(t)) - f(t, y(t)) \\ v_i(a) = u_i. \end{cases}$$

Note $v(a) = u$. Note also the preceding differential equality holds for almost all $t \in [a, b]$ and for all i. Since $y([a, b])$ is a compact subset of Ω, it follows there exist a neighborhood P of the origin in R^n and an integrable function $m : [a, b] \to R$ such that: $y(t) + P \subseteq \Omega$ for all $t \in [a, b]$;

$$\|f(t, y(t) + p) - f(t, y(t))\| \leq m(t)\|p\|$$

for almost all $t \in [a, b]$ and for all $p \in P$. We can suppose, taking a subsequence if necessary, that $s_i v_i(t) \in P$ for all $t \in [a, b]$ and for all i, hence $\|\dot{v}_i(t)\| \leq m(t)$ for almost all $t \in [a, b]$ and for all i, therefore the function v is absolutely continuous, and moreover, $\|\dot{v}(t)\| \leq m(t)$ for almost all $t \in [a, b]$.

In the following, $\langle z, w\rangle$ stands for the scalar product of the vectors $z \in R^n$ and $w \in R^n$.

Since for every $z \in R^n$ there hold the relations

$$\int_\alpha^\beta \liminf_{i\to\infty} \langle z, \dot{v}_i(\theta)\rangle \, d\theta \leq \lim_{i\to\infty} \int_\alpha^\beta \langle z, \dot{v}_i(\theta)\rangle \, d\theta =$$

$$= \lim_{i\to\infty} \langle z, v_i(\beta) - v_i(\alpha)\rangle = \langle z, v(\beta) - v(\alpha)\rangle = \int_\alpha^\beta \langle z, \dot{v}(\theta)\rangle \, d\theta$$

for all subintervals $[\alpha, \beta]$ of $[a, b]$, it follows

$$\liminf_{i\to\infty} \langle z, \dot{v}_i(t)\rangle \leq \langle z, \dot{v}(t)\rangle$$

for almost all $t \in [a, b]$.

Since R^n is separable, it follows that the preceding inequality holds for almost every $t \in [a, b]$ and for all $z \in R^n$. Since the multifunction $K_{f(t,\cdot)}(y(t))$ is a multidifferential for almost all $t \in [a, b]$, it follows

$$\inf \left\{ \langle z, w\rangle ; w \in K_{f(t,\cdot)}(y(t))(v(t)) \right\} \leq \liminf_{i\to\infty} \langle z, \dot{v}_i(t)\rangle$$

for almost all $t \in [a, b]$ and for all $z \in R^n$, hence $\dot{v}(t)$ belongs to the convex hull of the compact set $K_{f(t,\cdot)}(y(t))(v(t))$ for almost all $t \in [a, b]$. *QED*

Now, the question is whether the relation $v \in K_\sigma(x)(u)$ and the system in variation (2) are equivalent.

If the multidifferential $K_{f(t,\cdot)}(y(t))$ is point-valued for almost all $t \in [a, b]$, then the system in variation has a unique solution $v : [a, b] \to R^n$ (cf. [7, p. 193, ¶ 3]), hence also the multidifferential $K_\sigma(x)$ is point-valued and the equivalence does hold.

Otherwise the equivalence may fail. A counterexample is given next.

4 A counterexample

Let $f : [0, 1] \times R \to R$ be given by

$$f(x) = \begin{cases} x \sin(\ln(|x|)) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{3}$$

If $x \neq 0$, then $\dot{f}(x) = \sin(\ln(|x|)) \pm \cos(\ln(|x|))$, hence $|\dot{f}(x)| \leq \sqrt{2}$, therefore $|f(x_1) - f(x_2)| \leq \sqrt{2}|x_1 - x_2|$ for all $x_1 \in R$ and $x_2 \in R$.

In this case, the differential system (1) becomes

$$\begin{cases} \dot{y}(t) = \begin{cases} y(t) \sin(\ln(|y(t)|)) & \text{if } y(t) \neq 0 \\ 0 & \text{if } y(t) = 0, \end{cases} \\ y(0) = x. \end{cases}$$

The picture of the solutions to this differential system is elementary. If $f(x) = 0$ (which means either $x = 0$ or $|x| = \exp(\pm i\pi)$ for some natural number i), then $f(y(t)) = 0$ (namely $y(t) = x$) for all t. If $f(x) \neq 0$, then $f(y(t)) \neq 0$ for all t, hence $(\ln|y|)' = \sin(\ln|y|)$, and so

$$\ln\left|\tan\left(\frac{\ln|y(t)|}{2}\right)\right| - \ln\left|\tan\left(\frac{\ln|x|}{2}\right)\right| = t.$$

Now, let $x = 0$ and note $\sigma(0) = 0$ as well as $K_f(0)(u) = [-|u|, +|u|]$ for all $u \in R$. Further, let $u = 1$. Then the system in variation (2) becomes

$$\begin{cases} |\dot{v}(t)| \leq |v(t)|, \\ v(0) = 1. \end{cases}$$

Finally, let $v(t) = \exp(t)$ for all $t \in [0, 1]$. Then v is a solution to the system in variations, but $v \notin K_\sigma(0)(1)$.

Suppose, to the contrary, that $v \in K_\sigma(0)(1)$. Then there exists a sequence $(s_i, u_i, v_i) \in (0, +\infty) \times R \times \mathcal{C}([0,1]; R)$ which converges to $(0, 1, v)$ and which satisfies the relation $(1/s_i)\sigma(s_i u_i) = v_i$, hence

$$v_i(t) = u_i + \int_0^t (1/s_i) f(s_i v_i(\theta))\, d\theta.$$

We can suppose, taking a subsequence if necessary, that

$$\lim_{i\to\infty} (\sin(\ln(s_i)), \cos(\ln(s_i))) = (\sin(\ln(s)), \cos(\ln(s)))$$

for some $s \in [1, \exp(2\pi))$. Then

$$\lim_{i\to\infty} (1/s_i) f(s_i \xi_i) = (1/s) f(s\xi)$$

for every $\xi \neq 0$ and for every sequence ξ_i which converges to ξ, hence

$$v(t) = 1 + \int_0^t (1/s) f(sv(\theta))\, d\theta.$$

To conclude, $1 = \sin(\ln(s) + t)$ for all t, which is absurd.

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