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STABILITY OF MINIMALITY AND CRITICALITY IN DIRECTIONAL SET-VALUED OPTIMIZATION PROBLEMS

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by

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Abstract: In this paper we study stability issues for vectorial directional minima of sets and set-valued constrained optimization problems. In our work we consider several constructions and tools from interiority properties, enlargement of cones and the extremal principle to generalized Lipschitz properties. Our results complete the literature in this area of research, by proposing a different set of hypotheses for getting the stability of efficiency and preservation of criticality under perturbations.

Keywords: multiobjective optimization · directional minimality · stability · Painlevé-Kuratowski convergence

Mathematics Subject Classification (2010): 54C60 · 46G05 · 90C46

1 Introduction

In this paper we study some stability properties of directional Pareto minima. On the one hand, our study is a natural continuation of the investigation of the notions introduced and studied from the perspective of optimality conditions in [3], [7], and [2], and, on the other hand, our effort is partially inspired by several stability issues treated in [8] and [5]. More precisely, we consider the minima for sets and set-valued maps from a point of view that considers some special directions starting from the underlying minimum point instead of the classical approach that considers all the directions. This is motivated by some special classes of vector optimization problems such location problems (see [10]) and therefore, in view of possible practical applications, we are interested in a careful analysis of several types of directional solutions (strong, weak, approximate) in the sense of their stability under perturbations. Because of the special structure in terms of the constraints of the problems we consider, which cover, in particular, the classical case, the results we get generalize some similar assertions in literature. Furthermore, this very structure of the problems under consideration raise new technical issues that we try to solve without imposing too heavy assumptions.

In the case of directional Pareto minima for sets we consider only the problem of stability the efficiency points and, as in the general pattern often encountered in literature, such results need interiority assumptions. We discuss here such conditions from two perspectives: the nonemptiness of the topological interior of the ordering cone and a concept of enlargement (dilating) for a set of

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directions studied in [2], which allows to include the cone of constraint directions in the interior of a cone of directions.

For the case of vector optimization problems with set-valued maps, inspired by the results from the case of sets in terms of interiority assumptions, we are dealing with two types of results: stability of minimality and the preservation of criticality. In the results concerning the stability of minimality one takes a convergent sequence of minima for perturbed maps and one looks after the minimality of the limit for the initial, unperturbed map. In contrast, in the results concerning the preservation of the criticality one looks not necessary at the efficiency of the mentioned limit, but to the fact that this point is critical (in a generalized sense). Of course, criticality is weaker than efficiency and the second type of results allow us to relax some hypotheses and still retaining a valuable conclusion since it is known that even for algorithms the points of interest are the critical points. Nevertheless, in both kinds of results, we avoid some strong assumptions imposed in literature and we replace them with some metric properties.

We detail the organization of the paper and in this presentation we emphasize several aspects concerning the methods we employ in our work and the obtained results. The second section deals with notation and the main concept which are the starting points of our investigation. Then, we divide our work into two main parts. In the first part, which coincides with Section 3, we consider four concepts of directional Pareto minima for sets (whence we work on a single normed vector space) and we derive stability results for minima but as well for the constraints sets. One important ingredient is an enlargement procedure for which the main properties are discussed in [2]. The second part, that is Section 4, deals with vector optimization problems with set-valued maps and, naturally, two normed vector spaces are involved. We consider then a sequence of perturbed set-valued maps and we study conditions to ensure that a sequence of directional minima for these mappings converges to a minimum point or a critical point for the initial mapping. We devise several weak Lipschitz-type conditions that are enough to replace the role played in some works in literature by conditions involving the convergence of an intersection of sets. Furthermore, in the discussion concerning criticality, several methods issued from variational analysis and generalized differentiation calculus are employed. We illustrate our main results in both of the main sections of the paper by detailed examples. The paper ends with some concluding remarks where we emphasize some possible extension of our results.

2 Notation and preliminaries

Throughout this paper, we assume that X and Y are normed vector spaces over the real field $\mathbb{R}$ and on a product of normed vector spaces we consider the sum norm, unless otherwise stated. By $B(x,\varepsilon)$ we denote the open ball with center x and radius $\varepsilon > 0$ and by B_X the open unit ball of X. In the same manner, $D(x,\varepsilon)$ and D_X denote the corresponding closed balls. The symbol S_X stands for the unit sphere of X. By X^* we denote the topological dual of X, while w^* stands for the weak* topology on X^*.

Let $F : X \rightrightarrows Y$ be a set-valued map. As usual, the graph of F is

$$\operatorname{Gr} F := \{(x,y) \in X \times Y \mid y \in F(x)\},$$

and the inverse of F is the set-valued map $F^{-1} : Y \rightrightarrows X$ given by $(y,x) \in \operatorname{Gr} F^{-1}$ iff $(x,y) \in \operatorname{Gr} F$. Consider a nonempty subset A of X. Then the image of A through F is

$$F(A) := \{y \in Y \mid \exists x \in A : \ y \in F(x)\} = \bigcup_{x \in A} F(x).$$

and the distance function associated to A is $d_A : X \to \mathbb{R}$ given by

$$d_A(x) = d(x, A) := \inf_{a\in A} \|x - a\|.$$

The topological interior, topological closure, the convex hull, and conic hull of A are denoted, respectively, by $\operatorname{int} A$, $\operatorname{cl} A$, $\operatorname{conv} A$, $\operatorname{cone} A$. The negative polar of A is

$$A^- := \{x^* \in X^* \mid x^*(a) \le 0, \forall a \in A\}.$$

We will work on two different settings.

Firstly, we are going to consider minimality for sets in the vectorial (Pareto) sense and this means to work only in one space, namely X. Therefore, let $K \subset X$ be a proper (that is, $K \neq \{0\}$, $K \neq X$) convex and pointed cone. The positive dual cone of K is

$$K^+ := -K^- = \{x^* \in X^* \mid x^*(x) \ge 0, \forall x \in K\}$$

and it is well-known that K induces a partial order relation $\le_K$ on X by $x_1 \le_K x_2$ iff $x_2 - x_1 \in K$. If $\operatorname{int} K \neq \emptyset$, one can consider as well the strict partial order relation $<_K$ by $x_1 <_K x_2$ iff $x_2 - x_1 \in \operatorname{int} K$.

We recall the classical concepts. For a nonempty set $A \subset X$ an element $\overline{x} \in A$ is said to be a local Pareto minimum of A if there exists a neighborhood U of $\overline{x}$ such that $\overline{x}$ is a minimal element of $A \cap U$ with respect to the order given by K and this means

$$(A \cap U - \overline{x}) \cap -K = \{0\}.$$

If the cone K is solid (i.e., $\operatorname{int} K \neq \emptyset$), $\overline{x}$ is said to be local weak Pareto minimum of A if there exists a neighborhood U of $\overline{x}$ such that $\overline{x}$ is a minimal element of $A \cap U$ with respect to the strict order given by K, that is

$$(A \cap U - \overline{x}) \cap -\operatorname{int} K = \emptyset.$$

Secondly, we will deal with vectorial problems with set-valued objectives and for this we consider a pointed convex cone Q on Y which, as K before, characterizes a (strict) partial order relation on Y.

Take a set-valued mapping $F : X \rightrightarrows Y$, and let us consider the following geometrically constrained optimization problem with set-valued objective:

$$(P) \quad \text{minimize } F(x), \text{ subject to } x \in A,$$

where $A \subset X$ is a closed nonempty set.

The minimality is understood in the vectorial or Pareto sense as follows.

A point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (A \times Y)$ is a local Pareto minimum point for F on A if there exists a neighborhood U of $\overline{x}$ such that $\overline{x}$ is a Pareto minimum for $F(U \cap A)$, that is

$$(F(U \cap A) - \overline{y}) \cap -Q = \{0\}. \tag{2.1}$$

Suppose that $\operatorname{int} Q \neq \emptyset$. Similarly, the point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (A \times Y)$ is a local weak Pareto minimum point for F on A if there exists a neighborhood U of $\overline{x}$ such that $\overline{x}$ is a weak Pareto minimum of $F(U \cap A)$, that is

$$(F(U \cap A) - \overline{y}) \cap -\operatorname{int} Q = \emptyset. \tag{2.2}$$

The vectorial notions described by (2.1) and (2.2) cover as well the situation where f is a function (in which case $\overline{y} = f(\overline{x})$ will not be mentioned) and the situation of classical local minima in the scalar case (in which case we drop the label "Pareto").

Moreover, when $U = X$ we have the global minimality and we simply drop the word "local". In fact, this is the setting in which we develop the concepts under study. We will briefly refer again to the local case in the last section of the paper.

In fact, we will deal with some generalizations introduced in [3] of the concepts defined above and as mentioned, we do this in the next two sections.

3 Convergences of minima for sets

Besides the proper convex pointed and solid cone $K \subset X$, we consider as well a nonempty closed set $L \subset S_X$. Consider the next concepts.

Definition 3.1 *Let $A \subset X$ be a nonempty set $c \in K \setminus \{0\}$ and $\varepsilon > 0$, and $\overline{x} \in A$. One says that:*

(i) $\overline{x}$ is a directional Pareto minimum point for A with respect to L if

$$(A - \overline{x}) \cap \operatorname{cone} L \cap -K = \{0\}; \tag{3.1}$$

(ii) $\overline{x}$ is a weak directional Pareto minimum point for A with respect to L if

$$(A - \overline{x}) \cap \operatorname{cone} L \cap -\operatorname{int} K = \emptyset; \tag{3.2}$$

(iii) $\overline{x}$ is an $(\varepsilon, c)-$directional Pareto minimum point of A with respect to K if

$$(A - \overline{x} + \varepsilon c) \cap \operatorname{cone} L \cap -K = \emptyset;$$

(iv) $\overline{x}$ is an $(\varepsilon, c)-$weak directional Pareto minimum point of A with respect to K if

$$(A - \overline{x} + \varepsilon c) \cap \operatorname{cone} L \cap -\operatorname{int} K = \emptyset.$$

The concepts defined by (i) and (iii) are relevant only if $\operatorname{cone} L \cap -K \neq \{0\}$, while for (ii) and (iv) it is important to have $\operatorname{cone} L \cap -\operatorname{int} K \neq \emptyset$. Of course, (i) and (ii) above generalize the concept of global (weak) Pareto minimality. Furthermore (iii) and (iv) are directional generalization of approximate (weak) Pareto minimality (see [1], for instance).

Remark 3.2 *At a first look, in this setting where only the space X is considered, the notions of directional minimality defined above seem to be in fact minimality for the order induced by $-K \cap \operatorname{cone} L$. However, there are some natural reasons to consider these notions and some differences as well with respect to this simple reduction. First of all, one idea behind these concepts is that in one given (particular) problem the order is fixed by the cone K and then, if the minimality of a point with respect to that order is not verified, then we look at a set of directions that can be selected such that a partial form of minimality is fulfilled. On the other hand, for the weak counterparts, it is clear that $-K \cap \operatorname{cone} L$ is not solid, in general, and, even so, $\operatorname{cone} L \cap -\operatorname{int} K \neq \operatorname{int}(-K \cap \operatorname{cone} L)$.*

We denote the set of directional Pareto minimum points for A with respect to L by $\operatorname{DirMin}(A, L, K)$. Similarly, for the other three concepts in the previous definition we adopt the notation $\operatorname{WDirMin}(A, L, K)$, $(\varepsilon, c) - \operatorname{DirMin}(A, L, K)$ and $(\varepsilon, c) - \operatorname{WDirMin}(A, L, K)$, respectively.

Remark 3.3 *It is clear that if K is solid, then,*

$$\mathrm{DirMin}\,(A, L, K) \subset \mathrm{WDirMin}\,(A, L, K)\,,$$

and for every $\varepsilon > 0$ and for every $c \in K \setminus \{0\}$,

$$\begin{aligned} &\mathrm{WDirMin}\,(A, L, K) \subset (\varepsilon, c) - \mathrm{WDirMin}\,(A, L, K) \\ &(\varepsilon, c) - \mathrm{DirMin}\,(A, L, K) \subset (\varepsilon, c) - \mathrm{WDirMin}\,(A, L, K)\,. \end{aligned}$$

Notice that in these inclusions we take into account that $K + (0, \infty)\, c \subset K \setminus \{0\}$ and $\operatorname{int} K + [0, \infty) c \subset \operatorname{int} K$.

Example 3.4 The sets of directional minima introduced in Definition 3.1 are now illustrated with the following examples. We consider $X = \mathbb{R}^2, K = \operatorname{cone}\{(0,1)\,, (1,-1)\}$ and L consist of the points (x_1, x_2), on the unit circle, that lie in the third quadrant, together with the points on the axes. Let $\alpha > 0$ and A be the set

$$\left\{(x_1, 0) \in \mathbb{R}^2 \mid x_1 \in [-\alpha, 0)\right\} \cup \left\{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1 - 1)^2 + x_2^2 \leq 1\right\}.$$

Observe that $\operatorname{cone} L \cap -K = \operatorname{cone} L$ and $\operatorname{cone} L \cap - \operatorname{int} K = \left\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \leq 0, x_2 < 0\right\}$. Then we obtain that

$$\begin{aligned} \mathrm{DirMin}\,(A, L, K) &= \{(-\alpha, 0)\} \cup \left\{(x_1, -y_1) \in \mathbb{R}^2 \mid x_1 \in (0, 1]\right\} \\ &= \mathrm{WDirMin}\,(A, L, K)\,, \end{aligned}$$

where $y_1 = \sqrt{1 - (x_1 - 1)^2}$.

It is worth mentioning for instance that the set of directional Pareto minimum points isn't closed, even in the presence of closeness of the set A.

In what follows we consider α to be small enough. Let $\varepsilon > 0$ such that $\alpha < \varepsilon < 2\alpha$ and $c = (1, 0)$. We illustrate the geometry of approximate directional Pareto minima with the help of the figure below.

Thus $(\varepsilon, c) - \mathrm{WDirMin}\,(A, L, K)$ is the union of the segment from P_1 to $(0,0)$ and the plane region bounded by the path $P_2P_3P_4P_5P_2$ except the line segment P_3P_4 from which we remove the point P_4. On the other hand, the only points that ensure the nonemptiness of the set

$$(A - \overline{x} + \varepsilon c) \cap \operatorname{cone} L \cap -K$$

are in $(\varepsilon, c) - \mathrm{WDirMin}\,(A, L, K)$, except those of the path $P_2P_3P_4P_5$.

Let $A, (A_n)_{n \in \mathbb{N}}$ be nonempty subsets of X. We use the following notation:

$$\operatorname{Liminf} A_n = \{x \in X \mid \exists\, (x_n)\,, x_n \in A_n, \forall n \in \mathbb{N} : x_n \to x\}$$

and

$$\operatorname{Limsup} A_n = \{x \in X \mid \exists\, (n_k)\,, \exists\, (x_{n_k})\,, x_{n_k} \in A_{n_k}, \forall k \in \mathbb{N} : x_{n_k} \to x\}$$

Definition 3.5 *One says that A is the Painlevé-Kuratowski limit of (A_n) and notes $A_n \overset{P-K}{\to} A$ if the next conditions hold:*

$$A \subset \operatorname{Liminf} A_n \text{ and } \operatorname{Limsup} A_n \subset A.$$

Moreover, if the first of the above relations holds one writes $A_n \overset{P-K_-}{\to} A$, while if the second holds one writes $A_n \overset{P-K_+}{\to} A$.

From now on we suppose that $A, (A_n)_{n\in\mathbb{N}}$ are closed, unless otherwise stated.

We start with a lemma.

Lemma 3.6 *Suppose that X is a normed vector space. Let $\emptyset \neq L$ be a closed subset of S_X and $(L_n)_{n\in\mathbb{N}}$ be a sequence of nonempty and closed sets of S_X. If $L_n \overset{P-K_-}{\to} L$ then $\operatorname{cone} L_n \overset{P-K_-}{\to} \operatorname{cone} L$ and if $L_n \overset{P-K_+}{\to} L$, then $\operatorname{cone} L_n \overset{P-K_+}{\to} \operatorname{cone} L$.*

Proof. For the first part, take $a \in \operatorname{cone} L$. We have to find a sequence in $\operatorname{cone} L_n$ which approximate a. By definition of conic hull of L, there is $(\alpha, \ell) \in [0, \infty) \times L$ such that $a = \alpha\ell$. Since, by the assumption, $\ell \in L \subset \operatorname*{Liminf}_{n\to+\infty} L_n$, then for all n there is $\ell_n \in L_n$ such that $\ell_n \to \ell$. Hence the sequence $(\alpha\ell_n)_{n\in\mathbb{N}}$ is convergent to $\alpha\ell$ and since $\alpha\ell_n \in \operatorname{cone} L_n$ for all n, the conclusion follows.

For the second part, we have to prove that

$$\operatorname*{Limsup}_{n\to+\infty} \operatorname{cone} L_n \subset \operatorname{cone} L.$$

Let $x \in \operatorname*{Limsup}_{n\to+\infty} \operatorname{cone} L_n$. If $x = 0$ then clearly $x \in \operatorname{cone} L$. Suppose that $x \neq 0$. Based on the characterization with sequences of Limsup set, this means that there exists a subsequence $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ and $(a_{n_k})_{k\in\mathbb{N}}$ with $a_{n_k} \in \operatorname{cone} L_{n_k}$ for every $k \in \mathbb{N}$ such that $a_{n_k} \to x$. Then, for all $k \in \mathbb{N}$, there is $(\alpha_k, \ell_k) \subset [0, \infty) \times L_{n_k}$ such that $a_{n_k} = \alpha_k\ell_k$, for all k. But $\|a_k\| = \alpha_k$ for all k, so (α_k) is convergent to $\|x\| > 0$. Moreover, for k large enough, $\alpha_k \neq 0$. This means that $\ell_k = (\alpha_k)^{-1} \alpha_k\ell_k \to \|x\|^{-1} x$, so $\|x\|^{-1} x \in \operatorname*{Limsup}_{n\to+\infty} L_n \subset L$, and yields $x \in \operatorname{cone} L$. □

Proposition 3.7 *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of nonempty and closed subsets of X, $(L_n)_{n\in\mathbb{N}}$ be a sequence of nonempty and closed subsets of S_X. Consider $(x_n)_{n\in\mathbb{N}}$ be a sequence such that $x_n \in A_n$ for all $n \in \mathbb{N}$ and $x_n \to \overline{x} \in A$. If $A_n \overset{P-K_+}{\to} A$ and $L_n \overset{P-K_+}{\to} L$ for some $A \subset X$ and $L \subset S_X$, then*

$$A_n \cap [x_n + \operatorname{cone} L_n] \overset{P-K_+}{\to} A \cap (\overline{x} + \operatorname{cone} L).$$

Proof. We have to show that

$$\operatorname*{Limsup}_{n\to+\infty} (A_n \cap (x_n + \operatorname{cone} L_n)) \subset A \cap (\overline{x} + \operatorname{cone} L).$$

We start by fixing an arbitrary element $x \in \operatorname*{Limsup}_{n\to+\infty} (A_n \cap [x_n + \operatorname{cone} L_n])$. Then there exist a subsequence $(n_k)_{k\in\mathbb{N}}$ and $u_{n_k} \in A_{n_k} \cap (x_{n_k} + \operatorname{cone} L_{n_k})$ for every $k \in \mathbb{N}$ such that $u_{n_k} \to x$. From hypothesis we have that $\operatorname{Limsup} A_n \subset A$, hence $x \in A$. But, for all k

$$u_{n_k} - x_{n_k} \in \operatorname{cone} L_{n_k},$$

whence $x - \overline{x} \in \operatorname*{Limsup}_{n\to+\infty} L_n$ and by Lemma 3.6, $x - \overline{x} \in \operatorname{cone} L$. Therefore, $x \in A \cap (\overline{x} + \operatorname{cone} L)$. □

In the paper [2] a notion of enlargement that seems to be well adapted to the case of a cone generated by a given set of directions was introduced. We recall here this construction and its main properties.

Take $L \subset S_X$ and $\mu > 0$. Define

$$L_\mu := \{x \in S_X \mid d(x, L) \leq \mu\},$$

and we are looking for the cone generated by L_μ with respect to the cone generated by L. The next result is proved in [2].

Proposition 3.8 *Let $\emptyset \neq L \subset S_X$ and $\mu > 0$. Then:*

(i) L is closed if and only if $\operatorname{cone} L$ is closed;

(ii) L_μ is closed;

(iii) $\operatorname{cone} L \backslash \{0\} \subset \operatorname{int} \operatorname{cone} L_\mu$;

(iv) if L is closed, then

$$\bigcap_{\mu>0} L_\mu = L \text{ and } \bigcap_{\delta>0} \operatorname{cone} L_\delta = \operatorname{cone} L.$$

Remark 3.9 *As seen in the items (iii) and (iv), the cone generated by L_μ is a solid enlargement of $\operatorname{cone} L$.*

Using this type of enlargement we get some convergence results.

Proposition 3.10 *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of nonempty and closed subsets of X, $A \subset X$ a closed set, L a nonempty and closed subset of S_X, $\varepsilon > 0$ and $c \in K \backslash \{0\}$. Suppose that $A_n \overset{P-K_+}{\to} A$. Then for every $\mu > 0$*

$$\operatorname*{Limsup}_{n\to+\infty} (\varepsilon, c) - \operatorname{WDirMin}(A_n, L, K) \subset (\varepsilon, c) - \operatorname{WDirMin}(A, L_\mu, K).$$

Proof. Take an element $x \in \operatorname*{Limsup}_{n\to+\infty} (\varepsilon, c) - \operatorname{WDirMin}(A_n, L, K)$. We know that there exists a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ and $u_{n_k} \in \operatorname{WDirMin}(A_{n_k}, L, K)$ such that $u_{n_k} \to x$. In particular, $x \in \operatorname*{Limsup}_{n\to+\infty} A_n \subset A$. Suppose, by way of contradiction, that $x \notin (\varepsilon, c) - \operatorname{WDirMin}(A, L_\mu, K)$, i.e.,

$$(A - x + \varepsilon c) \cap \operatorname{cone} L \cap -\operatorname{int} K \neq \emptyset.$$

Therefore there is an element $v \in -\operatorname{int} K \cap \operatorname{cone} L$ such that $v \in A - x + \varepsilon c$. Then for all k,

$$0 \in A - x - v + \varepsilon c = A - u_{n_k} + \varepsilon c + u_{n_k} - x - v.$$

Notice that $u_{n_k} - x \to 0$. On the one hand $-v \in \operatorname{int} K$, so for k large enough, $u_{n_k} - x - v \in \operatorname{int} K$. On the other hand, $v \in \operatorname{cone} L \backslash \{0\}$ whence $v \in \operatorname{int} \operatorname{cone} L_\mu$. Again for all large k, $u_{n_k} - x - v \in -\operatorname{cone} L_\mu$ which means that we always have k such that

$$(A - u_{n_k} + \varepsilon c) \cap \operatorname{cone} L_\mu \cap -\operatorname{int} K \neq \emptyset.$$

This is a contradiction and we infer that x is a (ε, c) −weak directional Pareto minimum point of A with respect to L_μ. □

Proposition 3.11 *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of nonempty and closed subsets of X and $A \subset X$ be a nonempty closed set. Suppose that $A_n \overset{P-K_-}{\to} A$, let $\overline{x} \in A$ and $(x_n)_{n\in\mathbb{N}}$ a sequence of elements such that $x_n \in A_n$ for all n and $x_n \to \overline{x}$. Then for all $L \subset S_X$ and $\mu > 0$,*

$$A \cap (\overline{x} + \operatorname{cone} L) \subset \operatorname*{Liminf}_{n\to+\infty} (A_n \cap (x_n + \operatorname{cone} L_\mu)).$$

Proof. Let $x \in A \cap (\overline{x} + \operatorname{cone} L)$. If $x = \overline{x}$, then clearly x belongs to the right-hand side of the relation in the conclusion. Suppose that $x \neq \overline{x}$. Since $A_n \overset{P-K_-}{\to} A$ one can find for every $n \in \mathbb{N}$ an element $u_n \in A_n$ such that $u_n \to x$. Moreover $x - \overline{x} = \alpha\ell$, for some $\alpha > 0$ and $\ell \in L$. Observe that

$$\frac{u_n - x_n}{\|x - \overline{x}\|} \to \frac{x - \overline{x}}{\|x - \overline{x}\|} = \ell \in L.$$

According to the Proposition 3.8, we have for every n large enough

$$\frac{u_n - x_n}{\|x - \overline{x}\|} \in L_\mu,$$

whence $u_n - x_n \in \operatorname{cone} L_\mu$ for all n large enough. The conclusion follows. □

Proposition 3.12 *Suppose that* $\operatorname{int} K \neq \emptyset$. *Let* $A \subset X$ *be a closed set and take* $L \subset S_X$, $c \in K \backslash \{0\}$ *and* $\mu > 0$.

(i) Let $\varepsilon > 0$ *and* $(\delta_n)_{n\in\mathbb{N}} \subset (0, \varepsilon)$ *a sequence convergent to* ε. *If* (A_n) *is a sequence of closed subsets of* X *with* $A_n \overset{P-K_-}{\to} A$, *then*

$$\operatorname*{Limsup}_{n\to+\infty} (\delta_n, c) - \operatorname{WDirMin}(A_n, L_\mu, K) \subset (\varepsilon, c) - \operatorname{WDirMin}(A, L, K).$$

(ii) Consider $(\varepsilon_n) \subset (0, +\infty), \varepsilon_n \to 0$. *Then one has*

$$\operatorname*{Limsup}_{n\to+\infty} (\varepsilon_n, c) - \operatorname{WDirMin}(A, L_\mu, K) \subset \operatorname{WDirMin}(A, L, K).$$

Proof. (i) Take x arbitrarily from $\operatorname*{Limsup}_{n\to+\infty} (\delta, c) - \operatorname{WDirMin}(A_n, L_\mu, K)$. Then there exist a subsequence $(n_k)_{k\in\mathbb{N}}$ of the sequence of natural numbers and $u_{n_k} \in (\delta, c) - \operatorname{WDirMin}(A_n, L_\mu, K)$ for every $k \in \mathbb{N}$ such that $u_{n_k} \to x \in A$. Suppose that x is not an $(\varepsilon, c)-$weak directional Pareto minimum point of A with respect to L. This means that there exists $a \in A$ such that

$$a - x + \varepsilon c \in \operatorname{cone} L \cap - \operatorname{int} K.$$

Since $0 \notin \operatorname{cone} L \cap - \operatorname{int} K$, then there exists $\alpha > 0$ and $\ell \in L$ such that $a - x + \varepsilon c = \alpha\ell$. By the assumption $A_n \overset{P-K_-}{\to} A$, we can find a sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in A_n$ for all n and $a_n \to a$. Then,

$$a_{n_k} - u_{n_k} + \delta_{n_k} c = a - x + \varepsilon c + (a_{n_k} - a) - (u_{n_k} - x) + (\delta_{n_k} - \varepsilon) c$$

is, for k large enough, an element from $-\operatorname{int} K$.

In order to obtain a contradiction it is sufficient to prove that $a_{n_k} - u_{n_k} + \delta_{n_k} c$ belongs to $\operatorname{cone} L_\mu$, again for every k large enough. Since $\delta_{n_k} \to \varepsilon$, we obtain that

$$\frac{a_{n_k} - u_{n_k} + \delta_{n_k} c}{\|a_{n_k} - u_{n_k} + \delta_{n_k} c\|} \to \frac{a - x + \varepsilon c}{\|a - x + \varepsilon c\|} \in L.$$

According to the Proposition 3.8, we have for every k large enough

$$\frac{a_{n_k} - u_{n_k} + \delta_{n_k} c}{\|a_{n_k} - u_{n_k} + \delta_{n_k} c\|} \in L_\mu,$$

and hence $a_{n_k} - u_{n_k} + \delta_{n_k} c \in \operatorname{cone} L_\mu$ for all k large enough, which is against the minimality of a.

(ii) Take $x \in \operatorname*{Limsup}_{n\to+\infty} (\varepsilon_n, c) - \text{WDirMin}\,(A, L_\mu, K)$ and suppose that $x \notin \text{WDirMin}\,(A, L, K)$. Then, according to the definition, one has that $a - x \in \operatorname{cone} L \cap -\operatorname{int} K$, for some $a \in A$. Whence there exist $\alpha > 0$ and $\ell \in L$ such that $a - x = \alpha\ell$. For k large enough

$$\frac{a - x + \varepsilon_n c}{\|a - x + \varepsilon_n c\|} \to \frac{a - x}{\|a - x\|} = \ell.$$

Since, moreover, $\varepsilon_n \to 0$, we obtain that for k large enough, $a - x + \varepsilon_n c \in \operatorname{cone} L \cap -\operatorname{int} K$ which is a contradiction. Therefore, the conclusion is true. □

Remark 3.13 *Notice that the results in this section generalize several results from [5] which, practically, correspond to the case $L = S_X$.*

4 Stability of efficiency and criticality for vector optimization problems

Next, we consider the notation used for describing the problem (P) above and we present some directional concepts of efficiency.

Definition 4.1 *Let $\varepsilon > 0$, $c \in Q \setminus \{0\}$, $\emptyset \neq L \subset S_X$, $\emptyset \neq A \subset X$, A closed, and $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (A \times Y)$.*

(i) $(\overline{x}, \overline{y})$ is said to be a directional minimum point for F on A with respect to L, and we denote $(\overline{x}, \overline{y}) \in \text{DirMin}\,(F, A, L, Q)$, if

$$(F\,(A \cap [\overline{x} + \operatorname{cone} L]) - \overline{y}) \cap -Q = \{0\}\,.$$

(ii) If Q is solid, $(\overline{x}, \overline{y})$ is said to be a weak directional minimum point for F on A with respect to L, and we denote $(\overline{x}, \overline{y}) \in \text{WDirMin}\,(F, A, L, Q)$, if

$$(F\,(A \cap [\overline{x} + \operatorname{cone} L]) - \overline{y}) \cap -\operatorname{int} Q = \emptyset.$$

(iii) $(\overline{x}, \overline{y})$ is said to be an $(\varepsilon, c)-$directional minimum point for F on A with respect to L, and we denote $(\overline{x}, \overline{y}) \in (\varepsilon, c) - \text{WDirMin}\,(F, A, L, Q)$, if

$$(F\,(A \cap [\overline{x} + \operatorname{cone} L]) + \varepsilon c - \overline{y}) \cap -Q = \emptyset.$$

(iv) If Q is solid, $(\overline{x}, \overline{y})$ is said to be an $(\varepsilon, c)-$weak directional minimum point for F on A with respect to L, and we denote $(\overline{x}, \overline{y}) \in (\varepsilon, c) - \text{WDirMin}\,(F, A, L, Q)$, if

$$(F\,(A \cap [\overline{x} + \operatorname{cone} L]) + \varepsilon c - \overline{y}) \cap -\operatorname{int} Q = \emptyset.$$

Once again, (i) and (ii) are directional generalizations of the global Pareto solutions (see Section 2), while (iii) and (iv) consider approximate solutions.

Notice that in the setting of vector optimization problems with objective mappings that we consider in this section the sets L and Q lay in different spaces.

In what follows, we present some results about the stability of the solutions with respect to the perturbations of the multifunction F or the perturbations of the set A.

Theorem 4.2 *Let $\mu > 0$ and $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. Consider $(A_n)_{n\in\mathbb{N}}$ a sequence of nonempty and closed sets, A a nonempty and closed set, $(x_n, y_n)_{n\in\mathbb{N}}$ a sequence of elements such that $(x_n, y_n) \in \operatorname{WDirMin}(F_n, A_n, L_\mu, Q)$ for all $n \in \mathbb{N}$. Suppose that:*

(i) $\operatorname{Gr} F_n \overset{P-K_-}{\to} \operatorname{Gr} F$ and $A_n \overset{P-K_-}{\to} A$;

(ii) For every $(x, y) \in \operatorname{Gr} F$, there exist a neighborhood V of y such that for all $x_n' \to x$ and $x_n'' \to x$, there exists $(\lambda_n) \subset (0, +\infty)$ with $\lambda_n \|x_n' - x_n''\| \to 0$ for $n \to +\infty$, with

$$F_n(x_n') \cap V \subset F_n(x_n'') + \lambda_n \|x_n' - x_n''\| D_Y,$$

for all n large enough;

(iii) the sequence $(x_n, y_n)_{n\in\mathbb{N}}$ is convergent to $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (A \times Y)$.

Then $(\overline{x}, \overline{y}) \in \operatorname{WDirMin}(F, A, L, Q)$.

Proof. Suppose, by way of contradiction, that the conclusion is not true. Consequently, there are $x \in A \cap [\overline{x} + \operatorname{cone} L]$ and $y \in F(x)$ such that

$$y - \overline{y} \in -\operatorname{int} Q.$$

Since $\operatorname{Gr} F_n \overset{P-K_-}{\to} \operatorname{Gr} F$, it follows that there exists a sequence $(x_n', y_n')_{n\in\mathbb{N}}$ convergent to (x, y), with $(x_n', y_n') \in \operatorname{Gr} F_n$ for all $n \in \mathbb{N}$. Again from (i), one can find a sequence $(x_n'')_{n\in\mathbb{N}}$ convergent to x, with $x_n'' \in A_n$ for all $n \in \mathbb{N}$.

But, using (iii) from Proposition 3.11, we have

$$x - \overline{x} \in \operatorname{cone} L \subset \operatorname{int} \operatorname{cone} L_\mu,$$

so for all n sufficiently large

$$x_n'' - x_n \in \operatorname{cone} L_\mu.$$

Now, applying (ii) at the point (x, y), there exist a neighborhood V of y such that there exists $(\lambda_n) \subset (0, +\infty)$ with $\lambda_n \|x_n' - x_n''\| \to 0$ for $n \to +\infty$ and

$$y_n' \in F_n(x_n') \cap V \subset F_n(x_n'') + \lambda_n \|x_n' - x_n''\| D_Y,$$

for all n large enough. It follows that one can find $y_n'' \in F(x_n'')$ such that

$$\|y_n'' - y_n'\| \leq \lambda_n \|x_n' - x_n''\|,$$

for all n sufficiently large. Passing to limit and using that $\lambda_n \|x_n' - x_n''\| \to 0$, we obtain that the sequence $(y_n'')_{n\in\mathbb{N}}$ converges to y. We conclude therefore that for all $n \in \mathbb{N}$ large enough

$$y_n'' \in F_n(A_n \cap [x_n + \operatorname{cone} L_\mu])$$

and $y_n'' - y_n \in -\operatorname{int} Q$, since $(y_n'' - y_n)_{n\in\mathbb{N}}$ converges to $y - \overline{y}$. Whence

$$(F_n(A_n \cap [x_n + \operatorname{cone} L_\mu]) - y_n) \cap -\operatorname{int} Q = \emptyset,$$

for all n large enough, the required contradiction. We get the conclusion. □

Remark 4.3 *If one takes $c \in K \setminus \{0\}$ and $\varepsilon > 0$ under the notation and assumptions of the above theorem, if $(x_n, y_n) \in (\varepsilon, c) - \operatorname{WDirMin}(F_n, A_n, L_\mu, K)$ for all $n \in \mathbb{N}$ then one can similarly prove that $(\overline{x}, \overline{y}) \in (\varepsilon, c) - \operatorname{WDirMin}(F, A, L, Q)$.*

We illustrate the above result by the following examples.

Example 4.4 Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $Q = \mathbb{R}_+$, and $A_n = A = \mathbb{R}^2$. Take

$$L = \mathbb{R}^2_- \cap S_X$$

and, for $\rho > 1$, we consider the enlargement

$$C_\rho := \left\{ \alpha\left(-1, \rho^{-1}\right) + \beta\left(\rho^{-1}, -1\right) \mid \alpha, \beta \in \mathbb{R}_+ \right\}$$

which is, in fact, the $\mu-$enlargement of the cone L with $\mu = \frac{1}{\sqrt{1+\rho^2}}$. For easy computation, we prefer to work C_ρ, but we keep in mind that, actually, $C_\rho = \operatorname{cone} L_\mu$ and $\rho \to 0$ when $\mu \to \infty$.

Firstly, take $F_n : \mathbb{R}^2 \rightrightarrows \mathbb{R}$ given by $F_n(a,b) = \frac{2}{n} - a - b$ for all nonzero natural numbers n. Then, it is clear that if one takes $(x_n, y_n) = \left(\left(\frac{1}{n}, \frac{1}{n}\right), 0\right) \in \operatorname{Gr} F_n$ then, $(x_n, y_n) \in \operatorname{WDirMin}(F_n, A_n, L_\mu, Q)$ for all $\rho > 1$. Indeed,

$$F_n\left(\frac{1}{n} + \left(-\alpha + \beta\rho^{-1}\right), \frac{1}{n} + \left(\alpha\rho^{-1} - \beta\right)\right) = \alpha - \beta\rho^{-1} - \alpha\rho^{-1} + \beta = (\alpha + \beta)\left(1 - \rho^{-1}\right) \geq 0,$$

for every $\alpha, \beta \geq 0$. On the other hand, we take $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}$ given by $F(a,b) = -a - b$ and $(\overline{x}, \overline{y}) \in ((0,0), 0) \in \operatorname{Gr} F$ and observe that $(\overline{x}, \overline{y}) \in \operatorname{WDirMin}(F, A, L, Q)$, whence the conclusion of the theorem is confirmed in this case (one can check that all the hypotheses hold).

Take now $F_n : \mathbb{R}^2 \rightrightarrows \mathbb{R}$ given by $F_n(a,b) = \frac{2}{n} - a - b + \left(a - \frac{1}{n}\right)\left(b - \frac{1}{n}\right)$ for all nonzero natural numbers n. Then, $(x_n, y_n) = \left(\left(\frac{1}{n}, \frac{1}{n}\right), 0\right) \in \operatorname{Gr} F_n$ is in $\operatorname{WDirMin}(F_n, A_n, L, Q)$, but it is not in $\operatorname{WDirMin}(F_n, A_n, L_\mu, Q)$, for any $\mu > 0$. Indeed,

$$\begin{aligned} F_n\left(\frac{1}{n} + \left(-\alpha + \beta\rho^{-1}\right), \frac{1}{n} + \left(\alpha\rho^{-1} - \beta\right)\right) &= \alpha - \beta\rho^{-1} - \alpha\rho^{-1} + \beta + \alpha\beta + \alpha\beta\rho^{-2} - \alpha^2\rho^{-1} - \beta^2\rho^{-1} \\ &= \alpha\left(1 - \rho^{-1} + \beta + \beta\rho^{-2} - \alpha\rho^{-1}\right) + \beta\left(\rho - 1 - \beta\right)\rho^{-1}, \end{aligned}$$

and for $\alpha = 0$ and $\beta > \rho - 1$, one has

$$F_n\left(\frac{1}{n} + \left(-\alpha + \beta\rho^{-1}\right), \frac{1}{n} + \left(\alpha\rho^{-1} - \beta\right)\right) < 0.$$

Consider now $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}$ given by $F(a,b) = -a - b + ab$ and $(\overline{x}, \overline{y}) \in ((0,0), 0) \in \operatorname{Gr} F$ and observe that $(\overline{x}, \overline{y}) \notin \operatorname{WDirMin}(F, A, L, Q)$ since, for instance, $F(-2, -3) > 0$. The other hypotheses of Theorem 4.2 hold. We give a detail concerning the fulfillment of (ii). Clearly, the linear part of F_n is Lipschitz. We have to deal with the nonlinear part which means to consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(a,b) = ab$. Take two sequences $\left(a_n^1, b_n^1\right)$, $\left(a_n^2, b_n^2\right)$ convergent to the same point (a,b). Denote by M the common constant of boundedness for $\left(a_n^1\right)$ and $\left(b_n^2\right)$. Then

$$\begin{aligned} \left|f\left(a_n^1, b_n^1\right) - f\left(a_n^2, b_n^2\right)\right| &= \left|a_n^1 b_n^1 - a_n^2 b_n^2\right| \leq \left|a_n^1\right| \left|b_n^1 - b_n^2\right| + \left|b_n^2\right| \left|a_n^1 - a_n^2\right| \\ &\leq M\left(\left|b_n^1 - b_n^2\right| + \left|a_n^1 - a_n^2\right|\right) \leq \sqrt{2} M \left\|\left(a_n^1, b_n^1\right) - \left(a_n^2, b_n^2\right)\right\|, \end{aligned}$$

so the required inequality holds.

Consequently, in Theorem 4.2 it is essential to have the enlargement L_μ in the assumptions.

Remark 4.5 *In general, the assumptions one finds in literature to ensure the fact that the limit point of a sequence of minima for perturbation maps is a minimum of the initial objective map are quite demanding and this can be seen as well in the paper [8]. For instance, in the quoted work, it is required a kind of Painlevé-Kuratowski inferior limit for an intersection sequence involving the objectives* (F_n) *and the constraints* (A_n) *: see [8, Definition 2.3 (ii)]. Here, we avoid the condition of convergence of the intersection of sets.*

Remark 4.6 *The condition (ii) we use in Theorem 4.2 is a generalized Lipschitz-type condition that involves all the perturbations for n large enough. It was shown in [5] that this condition is closely linked to the Painlevé-Kuratowski convergence of the images of perturbations. However, as seen as well in the example above, this is a rather weak condition and quite easy to check, since it involves two sequences convergent to the same point.*

The next result aims at further eliminate some assumptions from the above result and to get however some conclusion on the underlying point. The idea in the next results is that the limit of a sequence of minima is not necessarily itself a minimum, but a critical point. This is an important conclusion since, basically if shows that a limit of minimum points verifies a generalized Fermat necessary optimality condition.

Before proceeding any further, we recall some basic fact and references concerning the generalized differentiation calculus.

Given a nonempty subset S of a Banach space X and $x \in S$, then for every $\varepsilon \geq 0$, the set of $\varepsilon-$normals to S at x is defined by

$$\widehat{N}_\varepsilon(S,x) = \left\{ x^* \in X^* \mid \limsup_{u \xrightarrow{S} x} \frac{x^*(u-x)}{\|u-x\|} \leq \varepsilon \right\},$$

where $u \xrightarrow{S} x$ means that $u \to x$ and $u \in S$. We denote by $\widehat{N}(S,x)$ the set $\widehat{N}_0(S,x)$ and we call it the Fréchet normal cone to S at x.

The basic (or limiting, or Mordukhovich) normal cone to S at $\overline{x} \in S$ is defined by

$$N(S,\overline{x}) = \{x^* \in X^* \mid \exists \varepsilon_n \stackrel{(0,\infty)}{\longrightarrow} 0, x_n \xrightarrow{S} \overline{x}, x_n^* \xrightarrow{w^*} x^*, x_n^* \in \widehat{N}_{\varepsilon_n}(S,x_n), \forall n \in \mathbb{N}\}.$$

In the main results of the rest of this section, all the spaces involved are Asplund. In this setting, if $S \subset X$ is closed around $\overline{x}$, then the basic normal cone takes the following form:

$$N(S,\overline{x}) = \{x^* \in X^* \mid \exists x_n \xrightarrow{S} \overline{x}, x_n^* \xrightarrow{w^*} x^*, x_n^* \in \widehat{N}(S,x_n), \forall n \in \mathbb{N}\}.$$

One may use the normal cones to the graph of the multifunction $F : X \rightrightarrows Y$ at $(\overline{x},\overline{y}) \in \operatorname{Gr} F$ (otherwise the normal cones are empty) to introduce generalized differentiation constructions that we briefly recall further.

The Fréchet coderivative of F at $(\overline{x},\overline{y})$ is the set-valued $\widehat{D}^*F(\overline{x},\overline{y}) : Y^* \rightrightarrows X^*$ with the values

$$\widehat{D}^*F(\overline{x},\overline{y})(y^*) = \{x^* \in X^* \mid (x^*,-y^*) \in \widehat{N}(\operatorname{Gr} F,(\overline{x},\overline{y}))\},$$

and the basic coderivative of F at $(\overline{x},\overline{y})$ is the set-valued map $D^*F(\overline{x},\overline{y}) : Y^* \rightrightarrows X^*$ with the values

$$D^*F(\overline{x},\overline{y})(y^*) = \{x^* \in X^* \mid (x^*,-y^*) \in N(\operatorname{Gr} F,(\overline{x},\overline{y}))\}.$$

In the particular case when F is a function, f say, since $\overline{y} \in F(\overline{x})$ means $\overline{y} = f(\overline{x})$, we write $\widehat{D}^* f(\overline{x})$ for $\widehat{D}^* f(\overline{x}, \overline{y})$, and similarly for D^*.

In the convex case, i.e. S is a convex subset of X, the basic normal cone admit the following representation

$$N(S, \overline{x}) = \{x^* \in X^* \mid x^*(x - \overline{x}) \leq 0, \forall x \in S\}.$$

More precisely, $N(S, \overline{x})$ coincides with the negative polar of the tangent cone to S at $\overline{x}$.

If $S \subset X$ is closed around $\overline{x} \in S$, one says that S is sequentially normally compact ((SNC), for short) at $\overline{x}$ if

$$\left[x_n \overset{S}{\to} \overline{x}, \quad x_n^* \overset{w^*}{\to} 0, \quad x_n^* \in \widehat{N}(S, x_n)\right] \Rightarrow x_n^* \to 0.$$

If S is a closed convex cone, C say, the (SNC) property at 0 is equivalent to

$$\left[(x_n^*) \subset C^+, \quad x_n^* \overset{w^*}{\to} 0\right] \Rightarrow x_n^* \to 0.$$

In particular, if $\operatorname{int} C \neq \emptyset$, then C is (SNC) at 0.

Given the closed subsets $S_1, S_2, ..., S_k$ of the space X, where $k \geq 2$, one say that $S_1, S_2, \ldots, S_k$ are allied at $\overline{x} \in S_1 \cap S_2 \cap \ldots \cap S_k$ if for every $(x_{in}) \overset{S_i}{\to} \overline{x}, x_{in} \in \widehat{N}(S_i, x_{in})$, $i \in \overline{1,k}$, the relation $(x_{1n}^* + ... + x_{kn}^*) \to 0$ implies $(x_{in}^*) \to 0$ for every $i \in \overline{1,k}$. The concept of alliedness was introduced by Penot and his coauthors in [13] and [11] in order to get a calculus rule for the Fréchet normal cone to the intersection of sets. More precisely, if the subsets $S_1, ..., S_k$ are allied at $\overline{x}$, then there exists $r > 0$ such that, for every $\varepsilon > 0$ and every $x \in [S_1 \cap ... \cap S_k] \cap B(\overline{x}, r)$, there exist $x_i \in S_i \cap B(x, \varepsilon)$, $i \in \overline{1,k}$ such that

$$\widehat{N}(S_1 \cap ... \cap S_k, x) \subset \widehat{N}(S_1, x_1) + ... + \widehat{N}(S_k, x_k) + \varepsilon D_{X^*}.$$

In what follows we use the results concerning the theory of generalized differentiation built on these objects directly at the places we need them, without separate quotation.

Finally, we recall the property of Lipschitz-likeness of multifunctions. Let $F : X \rightrightarrows Y$ and $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. One says that F is Lipschitz-like around $(\overline{x}, \overline{y})$ if there are some neighborhoods U and V of $\overline{x}$ and $\overline{y}$, respectively and $\lambda \geq 0$ such that

$$F(x') \cap V \subset F(x'') + \lambda \|x' - x''\| D_Y,$$

for all $x', x'' \in U$.

Now, we are ready to formulate the first of the announced results.

Theorem 4.7 *Let $F : X \to Y$ and (F_n) be a sequence of set-valued maps between the Asplund spaces X and Y. Take $A \subset X$ and $L \subset S_X$ be nonempty closed sets and suppose that for all n, $(x_n, y_n) \in \operatorname{Gr} F_n \cap (A \times Y)$ is a weak directional Pareto minimum point for F_n on A with respect to L. Suppose that*

(i) $(x_n, y_n) \to (\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (A \times Y)$;

(ii) there exists $(\lambda_n)_{n \geq n_0} \subset (0, +\infty)$ with $\lambda_n \|x_n - \overline{x}\| \to 0$ for $n \to +\infty$, such that for all $n \geq n_0$;

$$F((\overline{x} + \operatorname{cone} L) \cap A) \subset F_n((x_n + \operatorname{cone} L) \cap A) + \lambda_n \|x_n - \overline{x}\| D_Y,$$

(iii) F is Lipschitz-like around $(\overline{x}, \overline{y})$;

(iv) Q is (SNC) at 0;

(v) the sets A and $\overline{x}+\operatorname{cone} L$ are allied at $\overline{x}$.

Then there exists $y^* \in Q^+ \setminus \{0\}$ such that

$$0 \in D^*F(\overline{x},\overline{y})(y^*) + N(A,\overline{x}) + N(\operatorname{cone} L, 0).$$

Proof. Since for all n the point $(x_n, y_n) \in \operatorname{Gr} F_n$ is a weak directional Pareto minimum point for F_n on A with respect to L, then for all $c \in \operatorname{int} Q$,

$$\operatorname{Gr} F_n \cap [A \cap (x_n + \operatorname{cone} L) \times (y_n - Q) - (0, c)] = \emptyset.$$

Indeed, in the contrary case, there would be some $(x,y) \in \operatorname{Gr} F_n$ with $x \in A \cap (x_n + \operatorname{cone} L)$ and $y - y_n \in -Q - c \subset -\operatorname{int} Q$, which is not possible.

We show that the system

$$\{\operatorname{Gr} F, A \cap (\overline{x} + \operatorname{cone} L), (\overline{x},\overline{y})\}$$

is extremal (see [12, Definition 2.1]). Fix $c \in \operatorname{int} Q$ and consider a sequence $(t_n) \subset (0,\infty)$ with

$$\begin{aligned}
&\lim t_n = \infty \\
&\lim t_n (\|y_n - \overline{y}\| + \lambda_n \|x_n - \overline{x}\|) = 0.
\end{aligned}$$

Notice that such a sequence does exist since $\lim (\|y_n - \overline{y}\| + \lambda_n \|x_n - \overline{x}\|) = 0$. Now, we show that for all n large enough

$$\operatorname{Gr} F \cap \left[((\overline{x} + \operatorname{cone} L) \cap A) \times (\overline{y} - Q) - \left(0, \frac{c}{t_n}\right) \right] = \emptyset,$$

which in particular implies that the above system is extremal.

Suppose, by way of contradiction, that is not the case. Then there exist a subsequence (n_k) and some sequences $a_k \in (\overline{x} + \operatorname{cone} L) \cap A$, $b_k \in \overline{y} - Q - \frac{c}{t_{n_k}}$, and $(a_k, b_k) \in \operatorname{Gr} F$ for all k. Since for all k large enough,

$$F((\overline{x} + \operatorname{cone} L) \cap A) \subset F_{n_k}((x_{n_k} + \operatorname{cone} L) \cap A) + \lambda_{n_k} \|x_{n_k} - \overline{x}\| D_Y,$$

we get the existence of $a_k' \in (x_{n_k} + \operatorname{cone} L) \cap A$, $b_k' \in F_{n_k}(a_k')$ and $u_k \in D_Y$ such that

$$b_k = b_k' + \lambda_{n_k} \|x_{n_k} - \overline{x}\| u_k.$$

Therefore, for all k one has

$$\begin{aligned}
b_k' &= b_k - \lambda_{n_k} \|x_{n_k} - \overline{x}\| u_k \in \overline{y} - Q - \frac{c}{t_{n_k}} - \lambda_{n_k} \|x_{n_k} - \overline{x}\| u_k \\
&= y_{n_k} + (\overline{y} - y_{n_k}) - Q - \frac{c}{t_{n_k}} - \lambda_{n_k} \|x_{n_k} - \overline{x}\| u_k \\
&= y_{n_k} - Q - \left(\frac{c}{t_{n_k}} + \lambda_{n_k} \|x_{n_k} - \overline{x}\| u_k + (y_{n_k} - \overline{y}) \right).
\end{aligned}$$

But, for k large enough,

$$t_{n_k} \|(y_{n_k} - \overline{y}) + \lambda_{n_k} \|x_{n_k} - \overline{x}\| u_k\| \le t_{n_k} (\|y_{n_k} - \overline{y}\| + \lambda_{n_k} \|x_{n_k} - \overline{x}\|) < d(c, \operatorname{bd} Q),$$

whence

$$c + t_{n_k} (\lambda_{n_k} \|x_{n_k} - \overline{x}\| u_k + (y_{n_k} - \overline{y})) \in \operatorname{int} Q.$$

Finally, the relations we got,

$$b_k' - y_{n_k} \in -\operatorname{int} Q,$$
$$b_k' \in F_{n_k}\left(a_k'\right) \subset F\left((x_{n_k} + \operatorname{cone} L) \cap A\right)$$

lead to a contradiction with the minimality of (x_{n_k}, y_{n_k}).

Thus for

$$A_1 := \operatorname{Gr} F$$

and

$$A_2 := [(\overline{x} + \operatorname{cone} L) \cap A] \times (\overline{y} - Q),$$

the system $\{A_1, A_2, (\overline{x}, \overline{y})\}$ is an extremal system in $X \times Y$. The rest of the proof closely follows the proof of [3, Theorem 3.20], but for the sake of completeness and because some of the arguments will be used as well further, we give the details.

Since $X \times Y$ is an Asplund space and the sets A_1 and A_2 are closed around $(\overline{x}, \overline{y})$, we can apply the approximate extremal principle to this system (see, [12, Theorem 2.20]). Therefore, for every $n \in \mathbb{N} \setminus \{0\}$, there exist $\left(x_n^1, y_n^1\right) \in \operatorname{Gr} F \cap D\left((\overline{x}, \overline{y}), \frac{1}{n}\right)$, $x_n^2 \in (\overline{x} + \operatorname{cone} L) \cap A \cap D\left(\overline{x}, \frac{1}{n}\right)$, $y_n^2 \in (\overline{y} - Q) \cap D\left(\overline{y}, \frac{1}{n}\right)$, $x_n^{1*} \in X^*, x_n^{2*} \in X^*, y_n^{1*} \in Y^*, y_n^{2*} \in Y^*$ such that

$$\left(x_n^{1*}, y_n^{1*}\right) \in \widehat{N}\left(\operatorname{Gr} F, \left(x_n^1, y_n^1\right)\right) + \frac{1}{n} D_{X^* \times Y^*},$$
$$x_n^{2*} \in \widehat{N}\left((\overline{x} + \operatorname{cone} L) \cap A, x_n^2\right) + \frac{1}{n} D_{X^*},$$
$$y_n^{2*} \in \widehat{N}\left(\overline{y} - Q, y_n^2\right) + \frac{1}{n} D_{Y^*} = -\widehat{N}\left(Q, \overline{y} - y_n^2\right) + \frac{1}{n} D_{Y^*}$$

and

$$x_n^{1*} + x_n^{2*} = 0, \ y_n^{1*} + y_n^{2*} = 0, \ \left\|\left(x_n^{1*}, y_n^{1*}\right)\right\| + \left\|\left(x_n^{2*}, y_n^{2*}\right)\right\| = 1. \tag{4.1}$$

Therefore, there exist $\left(u_n^{1*}, v_n^{1*}\right) \in \frac{1}{n} D_{X^* \times Y^*}$, $u_n^{2*} \in \frac{1}{n} D_{X^*}$ and $v_n^{2*} \in \frac{1}{n} D_{Y^*}$ such that

$$x_n^{1*} - u_n^{1*} \in \widehat{D}^* F\left(x_n^1, y_n^1\right)\left(v_n^{1*} - y_n^{1*}\right),$$
$$x_n^{2*} - u_n^{2*} \in \widehat{N}\left((\overline{x} + \operatorname{cone} L) \cap A, x_n^2\right),$$
$$y_n^{2*} - v_n^{2*} \in -\widehat{N}\left(Q, \overline{y} - y_n^2\right) \subset Q^+.$$

Using relation (4.1) we obtain that the sequences $\left(x_n^{1*}\right)$, $\left(x_n^{2*}\right)$, $\left(y_n^{1*}\right)$ and $\left(y_n^{2*}\right)$ are bounded, and since X and Y are Asplund spaces, there exist $x_1^* \in X^*$, $x_2^* \in X^*$, $y_1^* \in Y^*$ and $y_2^* \in Y^*$ such that $x_n^{1*} \xrightarrow{w^*} x_1^*$, $x_n^{2*} \xrightarrow{w^*} x_2^*$, $y_n^{1*} \xrightarrow{w^*} y_1^*$, $y_n^{2*} \xrightarrow{w^*} y_2^*$. Obviously, $x_1^* + x_2^* = 0$ and $y_1^* + y_2^* = 0$.

Now, if $y_1^* = 0$, then $y_2^* = 0$, whence $y_n^{2*} - v_n^{2*} \xrightarrow{w^*} 0$ and using the (SNC) assumption we have that $y_n^{2*} - v_n^{2*} \to 0$, whence $y_n^{2*} \to 0$, so $y_n^{1*} \to 0$. Taking into account that F is Lipschitz-like around $(\overline{x}, \overline{y})$ and using [12, Theorem 1.43], we obtain that $x_n^{1*} - u_n^{1*} \to 0$ and since $u_n^{1*} \to 0$, we have that $x_n^{1*} \to 0$. Using again (4.1) we obtain that $x_n^{2*} \to 0$, which contradicts the fact that $y_n^{2*} \to 0$ and $\left\|\left(x_n^{2*}, y_n^{2*}\right)\right\| = 1$. Hence $y_1^* \neq 0$. Moreover, since $y_1^* + y_2^* = 0$, $y_n^{2*} - v_n^{2*} \xrightarrow{w^*} y_2^*$, $y_n^{2*} - v_n^{2*} \subset Q^+$ and Q^+ is weakly-star closed, we obtain that $-y_1^* = y_2^* \in Q^+$.

Further, using the hypothesis (iii), for every n large enough, we get that there exist $l_n \in (\overline{x} + \operatorname{cone} L) \cap B\left(x_n^2, \frac{1}{n}\right)$, $a_n \in A \cap B\left(x_n^2, \frac{1}{n}\right)$ such that

$$x_n^{2*} \in \widehat{N}\left((\overline{x} + \operatorname{cone} L) \cap A, x_n^2\right) + \frac{1}{n} D_{X^*} \subset \widehat{N}\left(\overline{x} + \operatorname{cone} L, l_n\right) + \widehat{N}\left(A, a_n\right) + \frac{2}{n} D_{X^*},$$

whence, there exist $a_n^* \in \widehat{N}(A, a_n), l_n^* \in \widehat{N}(\overline{x} + \operatorname{cone} L, l_n)$ such that $a_n^* + l_n^* - x_n^{2*} \to 0$. Further, we prove that (a_n^*) or (l_n^*) is bounded. Suppose by contradiction that both sequences are unbounded. It follows that for every n, there is k_n sufficiently large such that

$$n < \min \left\{ \left\| a_{k_n}^* \right\|, \left\| l_{k_n}^* \right\| \right\}. \tag{4.2}$$

For simplicity, we denote the subsequences $\left(a_{k_n}^*\right)$, $\left(l_{k_n}^*\right)$ by (a_n^*), (l_n^*), respectively. Now, since $a_n^* \in \widehat{N}(A, a_n), l_n^* \in \widehat{N}(\overline{x} + \operatorname{cone} L, l_n)$ we obtain that

$$\begin{aligned}
&\frac{1}{n} a_n^* \in \widehat{N}(A, a_n), \\
&\frac{1}{n} l_n^* \in \widehat{N}(\overline{x} + \operatorname{cone} L, l_n) = \widehat{N}(\operatorname{cone} L, l_n - \overline{x}).
\end{aligned}$$

Since

$$\frac{1}{n} \|a_n^* + l_n^*\| \leq \frac{1}{n} \left\| a_n^* + l_n^* - x_n^{2*} \right\| + \frac{1}{n} \left\| x_n^{2*} \right\|,$$

we obtain that $\frac{1}{n}(a_n^* + l_n^*) \to 0$, so using again the hypothesis of alliedness we obtain that $\frac{1}{n} a_n^* \to 0$ and $\frac{1}{n} l_n^* \to 0$, which is in contradiction with relation (4.2). Consequently, we obtain that $(a_n^*), (l_n^*) \subset X^*$ are bounded, thus there exist $a^*, l^* \in X^*$ such that $a_n^* \xrightarrow{w^*} a^*$ and $l_n^* \xrightarrow{w^*} l^*$, so

$$x_2^* = a^* + l^* \in N(A, \overline{x}) + N(\operatorname{cone} L, 0).$$

Now, observe from above that $x_1^* \in D^* F(\overline{x}, \overline{y})(y_2^*)$, with $y_2^* \in Q^+ \setminus \{0\}$ and since $x_1^* + x_2^* = 0$, we get that

$$0 \in D^* F(\overline{x}, \overline{y})(y_2^*) + N(A, \overline{x}) + N(\operatorname{cone} L, 0)$$

with $y_2^* \in Q^+ \setminus \{0\}$, i.e., the conclusion. $\square$

Remark 4.8 *Observe that condition (ii) in Theorem 4.7 is a generalized Lipschitz condition that involves the initial set-valued map and the perturbation multifunctions.*

In [2, Definition 2.11] a notion of proper minimum with respect to a set of directions in the input space is proposed. We recall this concept here in an adaptation fitted to our setting here.

Definition 4.9 *Let $L \subset S_X$ be a nonempty closed set. One says that $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ is a $L-$directional proper Pareto minimum point for F if there exists a constants $\varepsilon > 0$ such that*

$$(F(\overline{x} + \operatorname{cone} L_\varepsilon) - \overline{y}) \cap -Q = \{0\}.$$

Proposition 4.10 *Let $(x_n, y_n) \in \operatorname{Gr} F_n$ for all n and $(x_n, y_n) \to (\overline{x}, \overline{y}) \in \operatorname{Gr} F$. Suppose that for all n, (x_n, y_n) is a $L-$directional proper Pareto minimum point for F with the same $\varepsilon > 0$. Assume that $\operatorname{Gr} F_n \overset{P-K_-}{\to} \operatorname{Gr} F$. Then for all $\delta \in (0, \varepsilon)$, and all $q_0 \in \operatorname{int} Q$, the system $\{\operatorname{Gr} F, (\overline{x} + \operatorname{cone} L_\delta) \times (\overline{y} - Q - q_0), (\overline{x}, \overline{y})\}$ is extremal.*

Proof. Since for all n, (x_n, y_n) is a $L-$directional proper Pareto minimum point for F with the constant $\varepsilon > 0$, then for all $c \in \operatorname{int} Q$,

$$\operatorname{Gr} F_n \cap [((x_n + \operatorname{cone} L_\varepsilon)) \times (\overline{y} - Q) - (0, c)] = \emptyset.$$

The justification for this is the same as before.

Fix $q_0 \in \operatorname{int} Q$. Suppose that there is a sequence $c_p \to 0$ with elements in $\operatorname{int} Q$ such that for a subsequence still denoted (c_p),

$$\operatorname{Gr} F \cap [((\overline{x} + \operatorname{cone} L_\delta)) \times (\overline{y} - Q - q_0) - (0, c_p)] \neq \emptyset.$$

Then for all p, there is $(x_p, y_p) \in \operatorname{Gr} F$, $x_p \in \overline{x} + \operatorname{cone} L_\delta$, $y_p \in (\overline{y} - Q) - c_p$. Since

$$\operatorname{Gr} F \subset \operatorname{Liminf} \operatorname{Gr} F_n,$$

for all p there is a sequence $\left(x_p^n, y_p^n\right)_n$ with $\left(x_p^n, y_p^n\right) \in \operatorname{Gr} F_n$ for all n and p, such that $\left(x_p^n, y_p^n\right) \to (x_p, y_p)$. Then, for n large enough

$$\begin{aligned} x_p^n &= x_n + x_p^n - x_n \\ &= x_n + x_p^n - x_p + x_p - x_n \\ &\in x_n + x_p^n - x_p - x_n + \overline{x} + \operatorname{cone} L_\delta \\ &\subset x_n + \operatorname{cone} L_\varepsilon, \end{aligned}$$

and

$$\begin{aligned} y_p^n &= y_n + y_p^n - y_n \\ &= y_n + y_p^n - y_p + y_p - y_n \\ &\in y_n + y_p^n - y_p + y_p + (\overline{y} - Q - q_0) - c_p - y_n \\ &\subset y_n + \left(y_p^n - y_p\right) + (\overline{y} - y_n) - Q - q_0 - c_p \\ &\subset y_n - Q - c_p. \end{aligned}$$

This is a contradiction. The conclusion follows. $\square$

Proposition 4.11 *Let X, Y be Asplund spaces. Let $(x_n, y_n) \in \operatorname{Gr} F_n$ for all n and $(x_n, y_n) \to (\overline{x}, \overline{y}) \in \operatorname{Gr} F$. Suppose that*

(i) for all n, (x_n, y_n) is a $L-$directional proper Pareto minimum point for F with the same $\varepsilon > 0$;

(ii) $\operatorname{Gr} F_n \overset{P-K_-}{\to} \operatorname{Gr} F$;

(iii) F is Lipschitz-like around $(\overline{x}, \overline{y})$;

(iv) K is (SNC) at 0.

Then for all $\delta \in (0, \varepsilon)$, there is $y^* \in Q^+ \backslash \{0\}$ such that

$$0 \in D^* F(\overline{x}, \overline{y})(y^*) + N(\operatorname{cone} L_\delta, 0).$$

Proof. According to the preceding proposition, for all $\delta \in (0, \varepsilon)$, and all $q_0 \in \operatorname{int} Q$, the system

$$\{\operatorname{Gr} F, (\overline{x} + \operatorname{cone} L_\delta) \times (\overline{y} - Q - q_0), (\overline{x}, \overline{y})\}$$

is extremal. Then since $X \times Y$ is an Asplund space, as before, by applying the approximate extremal principle to this system, for every $n \in \mathbb{N} \backslash \{0\}$, there exist $\left(x_n^1, y_n^1\right) \in \operatorname{Gr} F \cap D\left((\overline{x}, \overline{y}), \frac{1}{n}\right)$,

$x_n^2 \in (\overline{x} + \operatorname{cone} L_\delta) \cap D\left(\overline{x}, \frac{1}{n}\right)$, $y_n^2 \in (\overline{y} - Q - q_0) \cap D\left(\overline{y}, \frac{1}{n}\right)$, $x_n^{1*} \in X^*, x_n^{2*} \in X^*, y_n^{1*} \in Y^*, y_n^{2*} \in Y^*$ such that

$$\begin{aligned}
&\left(x_n^{1*}, y_n^{1*}\right) \in \widehat{N}\left(\operatorname{Gr} F, \left(x_n^1, y_n^1\right)\right) + \frac{1}{n} D_{X^* \times Y^*},\\
&x_n^{2*} \in \widehat{N}\left(\overline{x} + \operatorname{cone} L_\delta, x_n^2\right) + \frac{1}{n} D_{X^*},\\
&y_n^{2*} \in \widehat{N}\left(\overline{y} - Q - q_0, y_n^2\right) + \frac{1}{n} D_{Y^*} = -\widehat{N}\left(Q + q_0, \overline{y} - y_n^2\right) + \frac{1}{n} D_{Y^*}
\end{aligned}$$

and

$$x_n^{1*} + x_n^{2*} = 0,\ y_n^{1*} + y_n^{2*} = 0,\ \left\|\left(x_n^{1*}, y_n^{1*}\right)\right\| + \left\|\left(x_n^{2*}, y_n^{2*}\right)\right\| = 1.$$

Therefore, there exist $\left(u_n^{1*}, v_n^{1*}\right) \in \frac{1}{n} D_{X^* \times Y^*}$, $u_n^{2*} \in \frac{1}{n} D_{X^*}$ and $v_n^{2*} \in \frac{1}{n} D_{Y^*}$ such that

$$\begin{aligned}
&x_n^{1*} - u_n^{1*} \in \widehat{D}^* F\left(x_n^1, y_n^1\right)\left(v_n^{1*} - y_n^{1*}\right),\\
&x_n^{2*} - u_n^{2*} \in \widehat{N}\left(\overline{x} + \operatorname{cone} L_\delta, x_n^2\right),\\
&y_n^{2*} - v_n^{2*} \in -\widehat{N}\left(Q + q_0, \overline{y} - y_n^2\right).
\end{aligned}$$

But, since $Q + q_0$ is a convex set, from [12, Proposition 1.3],

$$\begin{aligned}
\widehat{N}\left(Q + q_0, \overline{y} - y_n^2\right) &= \left\{y^* \in Y^* \mid \left\langle y^*, q + q_0 - \left(\overline{y} - y_n^2\right)\right\rangle \le 0,\ \forall q \in Q\right\}\\
&= \left\{y^* \in Y^* \mid \langle y^*, q\rangle + \langle y^*, q_0\rangle \le \left\langle y^*, \left(\overline{y} - y_n^2\right)\right\rangle\ \forall q \in Q\right\}\\
&\subset Q^+,
\end{aligned}$$

where for the last inclusion one uses that Q is a cone.

Clearly, the sequences $\left(x_n^{1*}\right)$, $\left(x_n^{2*}\right)$, $\left(y_n^{1*}\right)$ and $\left(y_n^{2*}\right)$ are bounded, and consequently, there exist $x_1^* \in X^*$, $x_2^* \in X^*$, $y_1^* \in Y^*$ and $y_2^* \in Y^*$ such that $x_n^{1*} \xrightarrow{w^*} x_1^*$, $x_n^{2*} \xrightarrow{w^*} x_2^*$, $y_n^{1*} \xrightarrow{w^*} y_1^*$, $y_n^{2*} \xrightarrow{w^*} y_2^*$ and $x_1^* + x_2^* = 0$, $y_1^* + y_2^* = 0$.

Using the same reasoning as in the proof of Theorem 4.7, one gets that $-y_1^* = y_2^* \in Q^+ \setminus \{0\}$. Then passing to the limit we get

$$0 \in D^* F\left(\overline{x}, \overline{y}\right)\left(-y_1^*\right) + N\left(\operatorname{cone} L_\delta, 0\right).$$

and this is the conclusion. □

5 Concluding remarks

All the results obtained are in the global case, but of course the concepts at the beginning of Sections 3 and 4 can be easily defined in a local setting. However, local counterparts of many of the above results can be stated, but nevertheless not all arguments are working properly, without further specific assumptions. The interested reader can adapt the proofs from the global setting into the local setting, and the main problem is to ensure the existence of a ball around $\overline{x}$ (for the issues in Section 3) or around $(\overline{x}, \overline{y})$ (for the problems in Section 4) where the underlying point enjoys the desired properties. It is a simple matter to see that this could be ensured, in general, if one asks that $\liminf \delta_n > 0$, where δ_n are the radii of the balls around the points in the sequence of minima of the perturbed problem where the envisaged efficiency property holds.

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