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Solutions for nonlinear Fokker-Planck equations with measures as initial data and McKean-Vlasov equations

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Solutions for nonlinear Fokker–Planck equations with measures as initial data and McKean-Vlasov equations

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Abstract

One proves the existence and uniqueness of a generalized (mild) solution for the nonlinear Fokker–Planck equation (FPE)

$$
\begin{aligned}
&u_t - \Delta(\beta(u)) + \operatorname{div}(D(x)b(u)u) = 0, \quad t \geq 0,\ x \in \mathbb{R}^d,\ d \neq 2,\\
&u(0,\cdot) = u_0, \text{in } \mathbb{R}^d,
\end{aligned}
$$

where $u_0 \in L^1(\mathbb{R}^d)$, $\beta \in C^2(\mathbb{R})$ is a nondecreasing function, $b \in C^1$, bounded, $b \geq 0$, $D \in (L^2 \cap L^\infty)(\mathbb{R}^d;\mathbb{R}^d)$ with $\operatorname{div} D \in L^\infty(\mathbb{R}^d)$, and $\operatorname{div} D \geq 0$, β strictly increasing, if b is not constant. Moreover, $t \to u(t,u_0)$ is a semigroup of contractions in $L^1(\mathbb{R}^d)$, which leaves invariant the set of probability density functions in $\mathbb{R}^d$. If $\operatorname{div} D \geq 0$, $\beta'(r) \geq a|r|^{\alpha-1}$, and $|\beta(r)| \leq Cr^\alpha$, $\alpha \geq 1$, $d \geq 3$, then $|u(t)|_{L^\infty} \leq Ct^{-\frac{d}{d+(\alpha-1)d}}\,|u_0|^{\frac{2}{2+(m-1)d}}$, $t > 0$, and the existence extends to initial data u_0 in the space $\mathcal{M}_b$ of bounded measures in $\mathbb{R}^d$. The solution map $\mu \mapsto S(t)\mu$, $t \geq 0$, is a Lipschitz contractions on $\mathcal{M}_b$ and weakly continuous in $t \in [0,\infty)$. As a consequence for arbitrary initial laws, we obtain weak solutions to a class of McKean-Vlasov SDEs with coefficients which have singular dependence on the time marginal laws.

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1 Introduction

Here, we consider the nonlinear parabolic equation (FPE)

$$\begin{aligned} &u_t - \Delta(\beta(u)) + \operatorname{div}(Db(u)u) = 0, \ \forall (t,x) \in [0,\infty) \times \mathbb{R}^d, \\ &u(0,\cdot) = \mu, \ \text{in } \mathbb{R}^d, \end{aligned} \tag{1.1}$$

where μ is a bounded measure on $\mathbb{R}^d$ and the functions $\beta : \mathbb{R} \to \mathbb{R}$, $D : \mathbb{R}^d \to \mathbb{R}^d$, $b : \mathbb{R} \to \mathbb{R}$, are assumed to satisfy the following hypotheses

(i) $\beta : \mathbb{R} \to \mathbb{R}$ *is a monotonically nondecreasing* C^1*-function,* $\beta(0) = 0$.

(ii) $D \in (L^2 \cap L^\infty)(\mathbb{R}^d; \mathbb{R}^d)$, $\operatorname{div} D \in L^\infty(\mathbb{R}^d)$.

(iii) $b \in C^1(\mathbb{R})$, b *bounded, nonnegative, and* $b \equiv \text{const.}$, *if* $(\operatorname{div} D)^- \not\equiv 0$, *or* β *is not strictly increasing.*

In statistical physics, the nonlinear Fokker–Planck equation (NFPE) (1.1) models anomalous diffusion processes. (See, e.g., [17].) As a matter of fact, (1.1) is an extension of the classical Smoluchowski equation

$$u_t = \operatorname{div}(\sigma \nabla u) + \operatorname{div}(Du),$$

where the linear diffusion term is replaced by a nonlinear one of the form $\beta'(u)\nabla u$ to model short-range interactions in anomalous media. In such a situation, u represents the physical density instead of the probability density as in the case of the Smoluchowski equation.

It should be mentioned also that, as in the classical linear theory, NFPE (1.1) is related to the McKean-Vlasov stochastic differential equation (see [18])

$$\begin{aligned} &dX(t) = D(X(t))b(u(t,X(t)))dt + \tfrac{1}{\sqrt{2}} \left(\tfrac{\beta(u(t,X(t)))}{u(t,X(t))} \right)^{\frac{1}{2}} dW(t), \ t \geq 0, \\ &X(0) = \xi_0, \end{aligned} \tag{1.2}$$

if $|\beta(r)| \le C|r|^m,\ m \ge 1$. More precisely, if u is a Schwartz distributional solution to (1.1) such that $u : [0,\infty) \to L^1(\mathbb{R}^d)$ is $\sigma(L^1, C_b)$-continuous with $u(0) = u_0\,dx$, $u_0 \in L^1 \cap L^\infty$, $u \ge 0$, and

$$\int_{\mathbb{R}^d} u(t,x)dx = 1,\ \forall t \ge 0,$$

then there exists a (probabilistically) weak solution process X to SDE (1.2) such that u is the probability density of its time marginal laws. (See [4]–[6].) Here we would like to note that there is a vast literature on McKean–Vlasov SDEs and on Fokker–Planck equations (see [15] and [11], in particular [11], Section 6.7 (iii), respectively, and the references therein). But the dependence of the coefficients on the solution in these references is much more regular than in our so called "Nemytskii case" (see [6], Section 2, for details).

The L^1-existence theory in $\mathbb{R}^d$ for Fokker–Planck equations of the form (1.1) in $\mathbb{R}^d$, $d \ge 1$, and weak existence for the corresponding McKean–Vlasov equations (1.2) was studied in [4]-[6], [7], via nonlinear semigroup theory in the Banach space $L^1(\mathbb{R}^d)$, $d \ge 1$. Here, we shall extend this approach to the case where the initial data is a bounded measure on $\mathbb{R}^d$. In this case, our existence result extends those in earlier work of H. Brezis and A. Friedman [14] and by M. Pierre [20], who studied the case $D \equiv 0$. One main ingredient of the proof is $L^1 - L^\infty$ regularity results for the solution of (1.1), extending fundamental work by Ph. Benilan [10], A. Pazy [19] and L. Veron [22].

For some technical reasons determined by the specific properties of the fundamental solution to the Laplace operator in $\mathbb{R}^d$, the analysis developed here is confined to the case where $d = 1$ or $d \ge 3$. One might suspects, however, that the case $d = 2$, which is singular and is not covered by the present approach, might be similarly treated, but this remains to be done.

A further contribution of this paper is its applications to McKean-Vlasov SDEs (see Section 6) which in turn was one motivation of this work.

The structure of the paper is as follows. Section 2 contains our main existence result for (1.1) for $\mu = u_0\,dx$, $u_0 \in L^1$. More precisely, Theorem 2.2 assures the existence of a generalized solution in the sense of Crandall-Liggett (see, e.g., [2] for an elaborate presentation) to equation (1.1), and establishes some of its properties, in particular, that this solution is also a weak solution to (1.1) in the sense of Schwartz distributions if $u_0 \in L^1 \cap L^\infty$. Section 3 is devoted to the proof of Theorem 2.2. In Section 4, we prove the $L^1 - L^\infty$ regularizing effects of the generalized solutions to (1.1) on the initial data (see Theorem 4.1) under the additional conditions that $\beta'(r) \ge a|r|^{\alpha-1}$,

$\alpha \geq 1$, $d \geq 3$, and $\operatorname{div} D \geq 0$. The existence of weak solutions to (1.1) for initial data from the space of bounded measures is contained in Theorem 5.2 and, finally, the applications to the construction of probabilistically weak solutions to McKean-Vlasov SDEs are contained in Section 6.

Notation. For each $1 \leq p \leq \infty$, $L^p(\mathbb{R}^d)$, briefly denoted L^p, is the space of Lebesgue integrable functions u on $\mathbb{R}^d$ with the standard norm $|\cdot|_p$. We denote by $W^{k,p}(\mathbb{R}^d)$ the Sobolev space of all functions in L^p with partial derivatives D_j^ℓ up to order k in L^p, where $D_j^\ell = \frac{\partial^\ell}{\partial x_j^\ell}$ is taken in the sense of Schwartz distributions $\mathcal{D}'(\mathbb{R}^d)$. Denote by $L^p_{\rm loc}$ the space of L^p-integrable functions on every compact set and $W^{k,p}_{\rm loc} = \{u \in L^p_{\rm loc};\ D_j^\ell u \in L^p_{\rm loc},\ 1 \leq j \leq d,\ \ell = 1, 2, ..., k\}$, $1 \leq p \leq \infty$. We set $H^1 = W^{1,2}(\mathbb{R}^d)$ and $H^2 = W^{2,2}(\mathbb{R}^d)$. We denote by $\langle \cdot, \cdot \rangle_2$ the scalar product of L^2 and by ${}_{H^{-1}}\langle \cdot, \cdot \rangle_{H^1}$ the duality functional between H^1 and its dual space $(H^1)' = H^{-1}$. $C_b(\mathbb{R}^d)$ is the space of all continuous and bounded functions $u : \mathbb{R}^d \to \mathbb{R}$ endowed with the supremum norm $\|\cdot\|_{C_b}$, and by $C_b^1(\mathbb{R}^d)$ the space of continuously differentiable bounded functions with bounded derivatives. By $\mathcal{M}_b(\mathbb{R}^d)$ we denote the space of bounded Radon measures on $\mathbb{R}^d$, that is, the dual of $C_b(\mathbb{R}^d)$. In the following, we shall simply denote $C_b(\mathbb{R}^d)$, $C_b^1(\mathbb{R}^d)$ by C_b, C_b^1, respectively, and $\mathcal{M}_b(\mathbb{R}^d)$ by $\mathcal{M}_b$. Denote by $\|\cdot\|_{\mathcal{M})b}$ the norm of $\mathcal{M}_b$, that is,

$$\|\mu\|_{\mathcal{M}_b} = \sup\{|\mu(\psi);\ |\psi|_{C_b} \leq 1\}.$$

We also set

$$\mathcal{P}_0(\mathbb{R}^d) = \left\{\rho \in L^1,\ \rho \geq 0,\ \int_{\mathbb{R}^d} \rho\, dx = 1\right\}.$$

By $C_0^\infty(\mathbb{R}^d)$, we denote the space of infinitely differentiable functions on $\mathbb{R}^d$ with compact support and use a similar notation on $(0, \infty) \times \mathbb{R}^d$.

For $1 < p < \infty$, we denote by $M^p(\mathbb{R}^d)$ (simply written M^p) the Marcinkiewicz space of all (classes of) measurable functions $u : \mathbb{R}^d \to \mathbb{R}$ such that

$$\|u\|_{M^p} = \min\left\{\lambda > 0; \int_K |u(x)| dx \leq \lambda(\text{meas } K)^{\frac{1}{p'}} \right. \\ \left. \text{for all Borel sets } K \subset \mathbb{R}^d\right\} < \infty,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. We recall that $M^p(\mathbb{R}^d) \subset L^q_{\rm loc}(\mathbb{R}^d)$ with continuous injection for $1 < q < p < \infty$.

Let $E(x) = \omega_d |x|_d^{2-d}$, $x \in \mathbb{R}^d$, $d \geq 3$, be the fundamental solution of the Laplace operator. (Here ω_d is the volume of the unit d-ball and $|\cdot|_d$ is the Euclidean norm of $\mathbb{R}^d$.) We recall that (see, e.g., [12]) that, for $d \geq 3$,

$$E \in M^{\frac{d}{d-2}}(\mathbb{R}^d),\ |\nabla E|_d \in M^{\frac{d}{d-1}}(\mathbb{R}^d), \tag{1.3}$$

and, for each $f \in L^1$, the solution $u \in L^1$ to the equation

$$-\Delta u = f \text{ in } \mathcal{D}'(\mathbb{R}^d)$$

is given by the convolution product $u = E * f$ and satisfies

$$\|u\|_{M^{\frac{d}{d-2}}} \leq \|E\|_{M^{\frac{d}{d-2}}} |f|_1. \tag{1.4}$$

and

$$\|\nabla u\|_{M^{\frac{d}{d-1}}} \leq \|\nabla E\|_{M^{\frac{d}{d-1}}} |f|_1. \tag{1.5}$$

2 Generalized solutions to NFPE (1.1)

We shall treat first FPE (1.1) for initial data $u_0 \in L^1$. Let $u_0 \in L^1$ be given.

Definition 2.1. A continuous function $u : [0, \infty] \to L^1$ is said to be a *generalized solution* (or *mild solution*) to equation (1.1) if, for each $T > 0$,

$$u(t) = \lim_{h \to 0} u_h(t) \text{ in } L^1 \text{ uniformly on each interval } [0, T], \tag{2.1}$$

where $u_h : [0, T] \to L^1$ is the step function, defined by the finite difference scheme

$$u_h(t) = u_h^i,\ \forall t \in (ih, (i+1)h],\ i = 0, 1, ..., N-1, \tag{2.2}$$

$$u_h^0 = u_0,\ \beta(u_h^i) \in L^1_{\rm loc},\ u_h^i \in L^1,\ \forall i = 1, 2, ..., N,\ Nh = T, \tag{2.3}$$

$$u_h^{i+1} - h\Delta(\beta(u_h^{i+1})) + h \operatorname{div}(Db(u_h^{i+1})u_h^{i+1}) = u_h^i, \text{ in } \mathcal{D}'(\mathbb{R}^d), \tag{2.4}$$

for all $i = 0, 1, ..., N-1$.

Theorem 2.2 is our first main result.

Theorem 2.2. *Let $d \neq 2$. Under Hypotheses* (i), (ii), (iii), *for each* $u_0 \in \overline{D(A)}$, *where* A *and* $\overline{D(A)}$ *are defined in* (3.10), (3.11) *below and* $\overline{D(A)}$ *is the* L^1*-closure of* $D(A)$, *there is a unique generalized solution* $u = u(t, u_0)$ *to equation* (1.1). *Now assume, in addition, that*

$$\beta \in C^2(\mathbb{R}). \tag{2.5}$$

Then $\overline{D(A)} = L^1$ *and, for every* $u_0 \in L^1 \cap L^\infty$,

$$|u(t)|_\infty \le \exp(|(\mathrm{div}\,\mathrm{D})^-|_\infty^{\frac{1}{2}} t)|u_0|_\infty, \ \forall t > 0. \tag{2.6}$$

If $u_0 \in \mathcal{P}_0(\mathbb{R}^d)$, *then*

$$u(t) \in \mathcal{P}_0(\mathbb{R}^d), \ \forall t \ge 0, \tag{2.7}$$

Moreover, $t \to S(t)u_0 = u(t, u_0)$ *is a strongly continuous semigroup of nonlinear contractions from* L^1 *to* L^1, *that is,* $S(t+s)u_0 = S(t)S(s)u_0$ *for* $0 < s < t$, *and*

$$|S(t)u_0 - S(t)\bar{u}_0|_1 \le |u_0 - \bar{u}_0|_1, \ \forall t > 0, \ u_0, \bar{u}_0 \in L^1. \tag{2.8}$$

If $u_0 \in L^1 \cap L^\infty$, *then* u *is a solution to* (1.1) *in the sense of Schwartz distributions on* $(0, \infty) \times \mathbb{R}^d$, *that is,*

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^d} (u(\varphi_t + b(u) D \cdot \nabla \varphi) + \beta(u) \Delta \varphi) dt\, dx \\ &\qquad + \int_{\mathbb{R}^d} u_0(x) \varphi(0, x) dx = 0, \ \forall \varphi \in C_0^\infty([0, \infty) \times \mathbb{R}^d). \end{aligned} \tag{2.9}$$

We also note that, by (2.6), equation (2.9) is well defined for all $\varphi \in C_0^\infty([0, \infty) \times \mathbb{R}^d)$.

Remark 2.3. In the case where $\Delta\beta(u)$ in (1.1) is replaced by $\mathrm{div}(A(u)\nabla u)$ with $r \mapsto A(r)$, $r \in \mathbb{R}$, a locally bounded map taking values in the symmetric nonnegative $d \times d$-matrices and the divergence term does not depend on x, there is a classical existence and uniqueness result in [16], however, in the sense of kinetic soslutions.

It should be noted that the uniqueness of the solution u given by Theorem 2.2 is claimed in the class of generalized solutions and not in that of distributional solutions. The latter is true in some special cases (see, e.g., [8], [9], [13]), but it is open in the general case we consider here.

3 Proof of Theorem 2.2

The idea of the proof is to associate with equation (1.1) an m-accretive operator A in L^1 and so to reduce (1.1) to the Cauchy problem

$$\begin{aligned} &\frac{du}{dt} + Au = 0, \quad t \geq 0, \\ &u(0) = u_0. \end{aligned}$$

To this purpose, consider in L^1 the nonlinear operator

$$\begin{aligned} A_0 y &= -\Delta\beta(y) + \mathrm{div}(Db(y)y), \ \forall y \in D(A_0), \\ D(A_0) &= \{y \in L^1, \ \beta(y) \in L^1_{\mathrm{loc}}, -\Delta\beta(y) + \mathrm{div}(Db(y)y) \in L^1\}, \end{aligned} \tag{3.1}$$

where the differential operators Δ and div are taken in $\mathcal{D}'(\mathbb{R}^d)$. The main ingredient of the proof is the following lemma.

Lemma 3.1. *We have*

$$R(I + \lambda A_0) = L^1, \ \forall \lambda > 0, \tag{3.2}$$

and there is an operator $J_\lambda : L^1 \to L^1$ *such that*

$$\begin{aligned} &J_\lambda u \in (I + \lambda A_0)^{-1} u, \qquad \forall u \in L^1, \\ &|J_\lambda u - J_\lambda v|_1 \leq |u - v|_1, \quad \forall u, v \in L^1, \lambda > 0. \end{aligned} \tag{3.3}$$

$$J_{\lambda_2} u = J_{\lambda_1}\left(\frac{\lambda_1}{\lambda_2} u + \left(1 - \frac{\lambda_1}{\lambda_2}\right) J_{\lambda_2} u\right), \ \forall\, 0 < \lambda_1, \lambda_2 < \infty. \tag{3.4}$$

Moreover,

$$\beta(J_\lambda u) \in L^q_{\mathrm{loc}}, \ 1 < q < \frac{d}{d-1}, \ \forall u \in L^1, \tag{3.5}$$

$$\begin{aligned} &|J_\lambda u|_\infty \leq (1 + |(\mathrm{div}\, D)^-|^{\frac{1}{2}}_\infty)|u|_\infty, \ \forall u \in L^1 \cap L^\infty, \\ &\qquad 0 < \lambda < \lambda_0 = ((|(\mathrm{div}\, D)^-|_\infty + |(\mathrm{div}\, D)^-|^{\frac{1}{2}}_\infty)|b|_\infty)^{-1}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} &|J_\lambda(u)|_\infty \leq C_\lambda |u|_\infty, \ \forall\, u \in L^1 \cap L^\infty \ \textit{ for some } C_\lambda \in (0, \infty) \\ &\qquad\qquad \textit{with } C_\lambda = 1, \textit{ if } \mathrm{div}\, D \geq 0, \end{aligned} \tag{3.7}$$

$$J_\lambda(\mathcal{P}_0(\mathbb{R}^d)) \subset \mathcal{P}_0(\mathbb{R}^d), \ \lambda > 0. \tag{3.8}$$

If (2.5) *holds, then*

$$|J_\lambda g - g|_1 \leq C\lambda \|g\|_{W^{2,2}(\mathbb{R}^d)}, \ \forall g \in C^\infty_0(\mathbb{R}^d). \tag{3.9}$$

Here $R(I + \lambda A_0)$ is the range of the operator $I + \lambda A_0$ and $(I + \lambda A_0)^{-1}$: $L^1 \to D(A_0)$ (which, in general, might be multivalues) is a right inverse of A_0. Before proving Lemma 3.1, let us discuss some consequences.

Define the operator $A : D(A) \subset L^1 \to L^1$,

$$Au = A_0 u, \quad \forall u \in D(A), \tag{3.10}$$

$$D(A) = \{J_\lambda v, \ v \in L^1\}. \tag{3.11}$$

It is easily seen by (3.4) that

$$D(A) = \{u = J_\lambda v, \ v \in L^1\}, \ \forall \ 0 < \lambda < \infty. \tag{3.12}$$

By Lemma 3.1, we have

Lemma 3.2.

(i) *J_λ coincides with the inverse $(I + \lambda A)^{-1}$ of $(I + \lambda A)$.*

(ii) *The operator A is m-accretive in L^1, that is, $R(I + \lambda A) = L^1$, $\forall \lambda > 0$, and*

$$|(I + \lambda A)^{-1} u - (I + \lambda A)^{-1} v|_1 \le |u - v|_1, \ \forall u, v \in L^1, \ \lambda > 0. \tag{3.13}$$

Moreover, (3.5), (3.6), (3.8) *hold with $(I + \lambda A)^{-1}$ instead of J_λ. If* (2.5) *holds, then $D(A)$ is dense in L^1.*

Proof of Lemma 3.2. (i) follows immediately by (3.10), (3.11), (3.12). Except for the density of $D(A)$ in L^1, all assertions of (ii) are immediate by the definition of A and Lemma 3.1. Now, assume that (2.5) holds and let us prove that $D(A)$ is dense in L^1 (that is, $\overline{D(A)} = L^1$). By (3.9), we have $C_0^\infty(\mathbb{R}^d) \subset \overline{D(A)}$ and, since $C_0^\infty(\mathbb{R}^d)$ is dense in L^1, the assertion follows.

We recall that Lemma 3.2 implies via the Crandall and Liggett theorem (see [2], p. 140) that, for each $u_0 \in \overline{D(A)} = L^1$ and $T > 0$, the Cauchy problem

$$\begin{aligned} &\frac{du}{dt} + Au = 0, \ t \in (0, T), \\ &u(0) = u_0, \end{aligned} \tag{3.14}$$

has a unique mild solution $u \in C([0, T]; L^1)$, that is,

$$u(t) = \lim_{h \to 0} u_h(t) \text{ in } L^1 \text{ uniformly on } [0, T], \tag{3.15}$$

where $u_h : [0,T] \to L^1$ is given by (2.2)-(2.4), that is,

$$\begin{aligned} & u_h(t) = u_h^{i+1},\ t \in (ih, (i+1)h], \\ & u_h^{i+1} + hAu_h^{i+1} = u_h^i,\ i = 0,1,...,N-1;\ Nh = T, \\ & u_h^0 = u_0. \end{aligned} \tag{3.16}$$

In fact, the solution $u = u(t, u_0)$ given by (3.15), (3.16) is given by the exponential formula

$$S(t)u_0 = u(t,u_0) = \lim_{n\to\infty} \left(I + \frac{t}{n}A\right)^{-n} u_0 \text{ in } L^1,\ t \ge 0, \tag{3.17}$$

where the convergence is uniform in t on compact intervals $[0,T]$, and $S(t)$ is a semigroup of contractions on L^1, that is,

$$\begin{gathered} S(t+s) = S(t)S(s)u_0,\ \forall t,s \ge 0, \quad S(0) = I, \\ |S(t)u_0 - S(t)\bar{u}_0|_1 \le |u - \bar{u}_0|_1,\ \forall u_0, \bar{u}_0 \in L^1,\ t \ge 0. \end{gathered}$$

Proof of Lemma 3.1. We shall follow the argument of [7] (Lemma 3.1). Namely, for $f \in L^1$ consider the equation

$$u + \lambda A_0 u = f \tag{3.18}$$

or, equivalently,

$$u - \lambda\Delta\beta(u) + \lambda \text{ div}(Db(u)u) = f \text{ in } \mathcal{D}'(\mathbb{R}), \tag{3.19}$$

$$u \in L^1,\ \beta(u) \in L^1_{\text{loc}}, -\Delta\beta(u) + \text{div}(Db(u)u) \in L^1. \tag{3.20}$$

We shall assume first $f \in L^1 \cap L^2$ and we approximate equation (3.19) by

$$u - \lambda\Delta\widetilde{\beta}_\varepsilon(u) + \lambda\varepsilon\widetilde{\beta}_\varepsilon(u) + \lambda \text{ div}(Db_\varepsilon(u)u) = f, \tag{3.21}$$

where $\widetilde{\beta}_\varepsilon(u) \equiv \beta_\varepsilon(u) + \varepsilon u$, and for $\varepsilon > 0,\ r \in \mathbb{R}$,

$$\beta_\varepsilon(r) \equiv \frac{1}{\varepsilon}\ (r - (I + \varepsilon\beta)^{-1}r) = \beta((I + \varepsilon\beta)^{-1}r), \tag{3.22}$$

and

$$b_\varepsilon(r) = \begin{cases} b, \text{ if } b \text{ is a constant}, & \\ \dfrac{(b * \rho_\varepsilon)(r)}{1 + \varepsilon|r|}, & \text{otherwise}. \end{cases}$$

Here $\rho_\varepsilon(r) = \frac{1}{\varepsilon}\ \rho\left(\frac{r}{\varepsilon}\right),\ \rho \in C_0^\infty(\mathbb{R}),\ \rho \geq 0$, is a standard modifier and by I we denote the identity on $\mathbb{R}$.

We are going to show that, for $\varepsilon \to 0$, the solution $\{u_\varepsilon\}$ to (3.21) is convergent to a solution u to (3.19). We can rewrite (3.21) as

$$(\varepsilon I - \Delta)^{-1}u + \lambda\widetilde{\beta}_\varepsilon(u) + \lambda(\varepsilon I - \Delta)^{-1}\mathrm{div}(Db_\varepsilon(u)u) = (\varepsilon I - \Delta)^{-1}f. \quad (3.23)$$

We set

$$\begin{array}{rcl} F_\varepsilon(u) &=& (\varepsilon I - \Delta)^{-1}u,\ G^1_\varepsilon(u) = \lambda\widetilde{\beta}_\varepsilon(u),\ u \in L^2, \\ G^2_\varepsilon(u) &=& \lambda(\varepsilon I - \Delta)^{-1}(\mathrm{div}(Db_\varepsilon(u)u)),\ u \in L^2. \end{array}$$

It is easily seen that F_ε and G^1_ε are accretive and continuous in L^2.

Since $r \mapsto b_\varepsilon(r)r$ is Lipschitz, we also have by assumptions (ii), (iii) that G^2_ε is continuous in L^2 and

$$\begin{array}{l} \displaystyle\int_{\mathbb{R}^d}(G^2_\varepsilon(u) - G^2_\varepsilon(\bar{u}))(u - \bar{u})dx \\ \displaystyle\qquad = -\lambda\int_{\mathbb{R}^d} D(b_\varepsilon(u)u - b_\varepsilon(\bar{u})\bar{u}) \cdot \nabla(\varepsilon I - \Delta)^{-1}(u - \bar{u})dx \\ \qquad \geq -C_\varepsilon\lambda|u - \bar{u}|_2|\nabla(\varepsilon I - \Delta)^{-1}(u - \bar{u})|_2,\ \forall u, \bar{u} \in L^2(\mathbb{R}^d). \end{array}$$

Moreover, we have

$$\int_{\mathbb{R}^d}(\varepsilon I - \Delta)^{-1}uu\, dx = \varepsilon|(\varepsilon I - \Delta)^{-1}u|_2^2 + |\nabla(\varepsilon I - \Delta)^{-1}u|_2^2,\ \forall u \in L^2.$$

Hence, for $u^* = u - \bar{u}$, we have

$$\begin{array}{l} (F_\varepsilon(u^*) + G^1_\varepsilon(u) - G^1_\varepsilon(\bar{u}) + G^2_\varepsilon(u) - G^2_\varepsilon(\bar{u}), u^*)_2 \\ \qquad \geq \lambda\varepsilon|u^*|_2^2 + |\nabla(\varepsilon I - \Delta)^{-1}u^*|_2^2 + \varepsilon|(\varepsilon I - \Delta)^{-1}u^*|_2^2 \\ \qquad -C_\varepsilon\lambda|u^*|_2|\nabla(\varepsilon I - \Delta)^{-1}u^*|_2. \end{array}$$

This implies that $F_\varepsilon + G^1_\varepsilon + G^2_\varepsilon$ is accretive and coercive on L^2 for $\lambda < \lambda_\varepsilon$, where $\lambda_\varepsilon > 0$ is sufficiently small. Since this operator is continuous and accretive, it follows that it is m-accretive and, therefore, surjective (because it is coercive). Hence, for each $f \in L^2 \cap L^1$ and $0 < \lambda < \lambda_\varepsilon$, equation (3.23) has a unique solution $u_\varepsilon \in L^2$ with $\widetilde{\beta}_\varepsilon(u_\varepsilon) \in H^1$. Since $\widetilde{\beta}_\varepsilon$ has a Lipschitz inverse, we have that $u_\varepsilon \in H^1$, and hence $b_\varepsilon(u_\varepsilon)u_\varepsilon \in H^1$, and so $\Delta\widetilde{\beta}_\varepsilon(u) \in L^2$ by (3.21).

We denote by $u_\varepsilon(f)\in H^1(\mathbb{R}^d)$ the solution to (3.23) for $f\in L^2\cap L^1$ and we shall prove that

$$|u_\varepsilon(f_1)-u_\varepsilon(f_2)|_1\le |f_1-f_2|_1,\ \forall f_1,f_2\in L^1\cap L^2. \tag{3.24}$$

To this purpose we set $u=u_\varepsilon(f_1)-u_\varepsilon(f_2)$, $f=f_1-f_2$. By (3.21), we have, for $u_i=u_\varepsilon(f_i)$, $i=1,2$,

$$\begin{aligned} u-\lambda\Delta(\widetilde{\beta}_\varepsilon(u_1)-\widetilde{\beta}_\varepsilon(u_2))+\varepsilon\lambda(\widetilde{\beta}_\varepsilon(u_1)-\widetilde{\beta}_\varepsilon(u_2))\\ +\lambda\,\mathrm{div}(D(b_\varepsilon(u_1)u_1-b_\varepsilon(u_2)u_2))=f. \end{aligned} \tag{3.25}$$

We consider the Lipschitzian function $\mathcal{X}_\delta:\mathbb{R}\to\mathbb{R}$,

$$\mathcal{X}_\delta(r)=\begin{cases} 1 & \text{for } r\ge\delta,\\ \dfrac{r}{\delta} & \text{for } |r|<\delta,\\ -1 & \text{for } r<-\delta, \end{cases} \tag{3.26}$$

where $\delta>0$. (We note that $\mathcal{X}_\delta(r)\to\mathrm{sign}\,r$ for $\delta\to 0$.) We set

$$\Phi_\varepsilon=\lambda\nabla(\widetilde{\beta}_\varepsilon(u_1)-\widetilde{\beta}_\varepsilon(u_2))-\lambda D(b_\varepsilon(u_1)u_1-b_\varepsilon(u_2)u_2)$$

and rewrite (3.25) as

$$u=\mathrm{div}\ \Phi_\varepsilon-\varepsilon\lambda(\widetilde{\beta}_\varepsilon(u_1)-\widetilde{\beta}_\varepsilon(u_2))+f.$$

We set $\Lambda_\delta=\mathcal{X}_\delta(\widetilde{\beta}_\varepsilon(u_1)-\widetilde{\beta}_\varepsilon(u_2))$. Since $\Lambda_\delta\in H^1(\mathbb{R}^d)$, it follows that $\Lambda_\delta\ \mathrm{div}\,\Phi_\varepsilon\in L^1$, and so we have

$$\begin{aligned} \int_{\mathbb{R}^d}u\Lambda_\delta dx&=-\int_{\mathbb{R}^d}\Phi_\varepsilon\cdot\nabla\Lambda_\delta dx\\ &\quad-\varepsilon\lambda\int_{\mathbb{R}^d}(\widetilde{\beta}_\varepsilon(u_1)-\widetilde{\beta}_\varepsilon(u_2))\Lambda_\delta dx+\int_{\mathbb{R}^d}f\Lambda_\delta dx\\ &=-\int_{\mathbb{R}^d}\Phi_\varepsilon\cdot\nabla(\widetilde{\beta}_\varepsilon(u_1)-\widetilde{\beta}_\varepsilon(u_2))\mathcal{X}_\delta'(\widetilde{\beta}_\varepsilon(u_1)\\ &\quad-\widetilde{\beta}_\varepsilon(u_2))dx-\varepsilon\lambda\int_{\mathbb{R}^d}(\widetilde{\beta}_\varepsilon(u_1)-\widetilde{\beta}_\varepsilon(u_2))\mathcal{X}_\delta(\widetilde{\beta}_\varepsilon(u_1)\\ &\quad-\widetilde{\beta}_\varepsilon(u_2))dx+\int_{\mathbb{R}^d}f\Lambda_\delta dx. \end{aligned} \tag{3.27}$$

We have

$$
\begin{aligned}
I_\delta^1 &= \int_{\mathbb{R}^d} D(b_\varepsilon(u_1)u_1 - b_\varepsilon(u_2)u_2)\cdot\nabla\Lambda_\delta dx \\
&= \frac{1}{\delta}\int_{[|\widetilde{\beta}_\varepsilon(u_1)-\widetilde{\beta}_\varepsilon(u_2)|\le\delta]} D(b_\varepsilon(u_1)u_1 - b_\varepsilon(u_2)u_2)\cdot\nabla(\widetilde{\beta}_\varepsilon(u_1)-\widetilde{\beta}_\varepsilon(u_2))dx.
\end{aligned}
$$

Since $D\in L^2(\mathbb{R}^d;\mathbb{R}^d)$ and since the inverse of $\widetilde{\beta}_\varepsilon$ is Lipschitz with constant $\frac{1}{\varepsilon}$, we have

$$
|b_\varepsilon(u_1)u_1 - b_\varepsilon(u_2)u_2| \le C_\varepsilon|u_1-u_2| \le \frac{C_\varepsilon}{\varepsilon}\,|\widetilde{\beta}_\varepsilon(u_1)-\widetilde{\beta}_\varepsilon(u_2)|.
$$

Therefore, it follows that

$$
\begin{aligned}
&\lim_{\delta\to0}\frac{1}{\delta}\int_{[|\widetilde{\beta}_\varepsilon(u_1)-\widetilde{\beta}_\varepsilon(u_2)|\le\delta]} |D(b_\varepsilon(u_1)u_1 - b_\varepsilon(u_2)u_2)\cdot\nabla(\widetilde{\beta}_\varepsilon(u_1)-\widetilde{\beta}_\varepsilon(u_2))|dx \\
&\qquad\le \frac{C_\varepsilon}{\varepsilon}\,|D|_2\lim_{\delta\to0}\left(\int_{[|\widetilde{\beta}_\varepsilon(u_1)-\widetilde{\beta}_\varepsilon(u_2)|\le\delta]}|\nabla(\widetilde{\beta}_\varepsilon(u_1)-\widetilde{\beta}_\varepsilon(u_2))|^2dx\right)^{\frac12} = 0, \quad (3.28)
\end{aligned}
$$

because $\nabla(\widetilde{\beta}_\varepsilon(u_1)-\widetilde{\beta}_\varepsilon(u_2))(x)=0$, a.e. on $[x\in\mathbb{R}^d;\widetilde{\beta}_\varepsilon(u_1(x))-\widetilde{\beta}_\varepsilon(u_2(x))=0]$. This yields

$$
\lim_{\delta\to0} I_\delta^1 = 0,
$$

Since $\mathcal{X}'_\delta\ge0$, a.e. on $\mathbb{R}$, we also have

$$
\int_{\mathbb{R}^d}\nabla(\widetilde{\beta}_\varepsilon(u_1)-\widetilde{\beta}_\varepsilon(u_2))\cdot\nabla(\widetilde{\beta}_\varepsilon(u_1)-\widetilde{\beta}_\varepsilon(u_2))\mathcal{X}'_\delta(\widetilde{\beta}_\varepsilon(u_1)-\widetilde{\beta}_\varepsilon(u_2))\,dx\ge0.
$$

Then, by (3.27), we get

$$
\lim_{\delta\to0}\int_{\mathbb{R}^d} u\mathcal{X}_\delta(\widetilde{\beta}_\varepsilon(u_1)-\widetilde{\beta}_\varepsilon(u_2))dx \le \int_{\mathbb{R}^d}|f|\,dx,\ \forall\varepsilon>0,
$$

and, since $u\mathcal{X}_\delta(\widetilde{\beta}_\varepsilon(u_1)-\widetilde{\beta}_\varepsilon(u_2))\ge0$ and $\mathcal{X}_\delta\to$ sign as $\delta\to0$, by Fatou's lemma this yields

$$
|u|_1\le|f|_1, \tag{3.29}
$$

as claimed. We note that, since $f=0$ implies $u_\varepsilon(f)=0$, it follows by (3.29) that $u_\varepsilon(f)\in L^1$, $\forall f\in L^1\cap L^2$.

Next, for f arbitrary in L^1, consider a sequence $\{f_n\} \subset L^2$ such that $f_n \to f$ strongly in L^1. Let $\{u_\varepsilon^n\} \subset L^1 \cap L^2$ be the corresponding solutions to (3.21) for $0 < \lambda < \lambda_\varepsilon$. Taking into account (3.29), we obtain by the above equation that

$$|u_\varepsilon^n - u_\varepsilon^m|_1 \le |f_n - f_m|_1,\ \forall n, m \in \mathbb{N}.$$

Hence, for $n \to \infty$, we have $u_\varepsilon^n \to u_\varepsilon(f)$ in L^1. Define the operator

$$\begin{aligned} A_\varepsilon u &= -\Delta\widetilde{\beta}_\varepsilon(u) + \varepsilon\widetilde{\beta}_\varepsilon(u) + \operatorname{div}(Db_\varepsilon(u)u) \\ D(A_\varepsilon) &= \{u \in L^1; -\Delta\widetilde{\beta}_\varepsilon(u) + \varepsilon\widetilde{\beta}(u) + \operatorname{div}(Db_\varepsilon(u)u) \in L^1\}. \end{aligned} \tag{3.30}$$

It is obvious that $(A_\varepsilon, D(A_\varepsilon))$ is closed on L^1. Therefore, $u_\varepsilon(f) \in D(A_\varepsilon)$ and

$$u_\varepsilon(f) + \lambda A_\varepsilon u_\varepsilon(f) = f, \tag{3.31}$$

for $\lambda < \lambda_\varepsilon$. We also have

$$|u_\varepsilon(f_1) - u_\varepsilon(f_2)|_1 \le |f_1 - f_2|_1,\ \forall f_1, f_2 \in L^1, \tag{3.32}$$

for $\lambda < \lambda_\varepsilon$.

Then, by Proposition 3.3 in [2], p. 99, it follows that $R(1 + \lambda A_\varepsilon) = L^1$, $\forall \lambda > 0$, and also (3.32) extends to all $\lambda > 0$ (see Proposition 3.1 in [2]).

Moreover, if $u_\varepsilon = u_\varepsilon(\lambda, f)$ is our solution to (3.21), we have, for all $0 < \lambda_1, \lambda_2 < \infty$ and $f \in L^1 \cap L^2$, by definition

$$u_\varepsilon(\lambda_2, f) = u_\varepsilon\left(\lambda_1, \frac{\lambda_1}{\lambda_2} f + \left(1 - \frac{\lambda_1}{\lambda_2}\right) u_\varepsilon(\lambda_2, f)\right). \tag{3.33}$$

If $f \in L^1 \cap L^\infty$, we have

$$|u_\varepsilon(f)|_\infty \le (1 + |(\operatorname{div} D)^-|_\infty^{\frac{1}{2}})|f|_\infty, \quad 0 < \lambda < \lambda_0. \tag{3.34}$$

Indeed, by (3.21), we see that, for $M \in [0, \infty)$, $u_\varepsilon = u_\varepsilon(f)$, $b_\varepsilon^*(r) = b_\varepsilon(r)r$ and $\lambda < \lambda_0$,

$$\begin{aligned} &(u_\varepsilon - |f|_\infty - M) - \lambda\Delta(\widetilde{\beta}_\varepsilon(u_\varepsilon) - \widetilde{\beta}_\varepsilon(|f|_\infty + M)) \\ &\quad + \lambda\varepsilon(\widetilde{\beta}_\varepsilon(u_\varepsilon) - \widetilde{\beta}_\varepsilon(|f|_\infty + M)) + \lambda \operatorname{div}(D(b_\varepsilon^*(u_\varepsilon) - b_\varepsilon^*(|f|_\infty + M))) \\ &\quad \le f - |f|_\infty - M - \lambda b_\varepsilon^*(M + |f|_\infty)\operatorname{div} D \le 0. \end{aligned}$$

Multiplying the above equation by $\mathcal{X}_\delta((u_\varepsilon-(|f|_\infty+M))^+)$ and integrating over $\mathbb{R}^d$, we get as above, for $\delta\to 0$,

$$|(u_\varepsilon-|f|_\infty-M)^+|_1\le 0$$

and, therefore, choosing $M=|(\operatorname{div}D)^-|_\infty^{\frac12}|f|_\infty$,

$$u_\varepsilon\le(1+|(\operatorname{div}D)^-|_\infty^{\frac12})|f|_\infty,\ \ \text{a.e. in }\mathbb{R}^d.$$

Similarly, one gets that

$$u_\varepsilon\ge-(1+|(\operatorname{div}D)^-|_\infty^{\frac12})|f|_\infty,\ \ \text{a.e. in }\mathbb{R}^d,$$

and so (3.34) follows, which in turn, by (3.33), implies that, for some $C_\lambda\in(0,\infty)$,

$$|u_\varepsilon(\lambda,f)|_\infty\le C_\lambda|f|_\infty \text{ for all } \varepsilon>0,\ f\in L^1\cap L^\infty, \tag{3.35}$$

where $C_\lambda=1$, if $\operatorname{div}D\ge 0$.

Now, we are going to let $\varepsilon\to 0$ in (3.21). To this end, we need some estimates on u_ε.

By (3.21) (or (3.31)), we have

$$\Delta\widetilde{\beta}_\varepsilon(u_\varepsilon)=\lambda^{-1}(u_\varepsilon-f)+\operatorname{div}(Db_\varepsilon(u_\varepsilon)u_\varepsilon)+\varepsilon\widetilde{\beta}_\varepsilon(u_\varepsilon). \tag{3.36}$$

We shall consider first the case $d\ge 3$. This yields

$$\widetilde{\beta}_\varepsilon(u_\varepsilon)=\frac{1}{\lambda}\,E*(-u_\varepsilon-\varepsilon\lambda\widetilde{\beta}_\varepsilon(u_\varepsilon)+f)+\nabla(E*(Db_\varepsilon(u_\varepsilon)u_\varepsilon)),\ \text{a.e. in }\mathbb{R}^d, \tag{3.37}$$

where E is the fundamental solution to the Laplace operator (see Section 1). Then, recalling (1.3), (1.4) and (3.29), we get, for $\varepsilon\in(0,1)$,

$$\begin{aligned}&\|\widetilde{\beta}_\varepsilon(u_\varepsilon)-\nabla(E*(Db_\varepsilon(u_\varepsilon)u_\varepsilon))\|_{M^{\frac{d}{d-2}}}\\ &\quad\le\frac{1}{\lambda}\,\|E\|_{M^{\frac{d}{d-2}}}|u_\varepsilon+\lambda\varepsilon\widetilde{\beta}_\varepsilon(u_\varepsilon)-f|_1\\ &\quad\le\frac{2+3\lambda}{\lambda}\,\|E\|_{M^{\frac{d}{d-2}}}|f|_1,\ \forall\lambda>0,\end{aligned} \tag{3.38}$$

because $\varepsilon|\widetilde{\beta}_\varepsilon(u)|\le(2+\varepsilon^2)|u|$. Taking into account that, for $d\ge 3$,

$$\begin{aligned}M^{\frac{d}{d-2}}&\subset L^p_{\rm loc},\ \ \forall p\in\left(1,\tfrac{d}{d-2}\right)\\ M^{\frac{d}{d-1}}&\subset L^p_{\rm loc},\ \ \forall p\in\left(1,\tfrac{d}{d-1}\right)\end{aligned}$$

we see by (3.38) that, for $1 < p < \frac{d}{d-2}$, and, for all compacts $K \subset \mathbb{R}^d$, we have

$$\|\widetilde{\beta}_\varepsilon(u_\varepsilon) - \nabla(E * (Db_\varepsilon(u_\varepsilon)u_\varepsilon))\|_{L^p(K)} \le \frac{1+\lambda}{\lambda}\, C_K|f|_1,\ \forall \lambda > 0,$$

and so, by (1.3), we have, for $1 < q < \frac{d}{d-1}$,

$$\begin{aligned}\|\widetilde{\beta}_\varepsilon(u_\varepsilon)\|_{L^q(K)} &\le C_K\left(\|\nabla(E * (Db_\varepsilon(u_\varepsilon)u_\varepsilon))\|_{L^q(K)} + \frac{1+\lambda}{\lambda}|f|_1\right)\\ &\le C_K\left(\|\nabla(E * (Db_\varepsilon(u_\varepsilon)u_\varepsilon))\|_{M^{\frac{d}{d-1}}} + \frac{1+\lambda}{\lambda}|f|_1\right)\\ &\le C_K\left(|u_\varepsilon|_1 + \frac{1+\lambda}{\lambda}|f|_1\right)\\ &\le C_K\left(\frac{1+\lambda}{\lambda}\right)|f|_1,\ \forall \lambda > 0,\end{aligned} \tag{3.39}$$

for any compact subset $K \subset \mathbb{R}^d$, where the constant C_K changes from line to line and we used that $|b_\varepsilon|_\infty \le |b|_\infty$.

We assume first that $f \in L^1 \cap L^\infty$and $0 < \lambda < \lambda_0$. Then, by (3.34), we have, along a subsequence $\{\varepsilon\} \to 0$,

$$u_\varepsilon \to u \text{ weak-star in } L^\infty, \text{ whence weakly in } L^1_{\rm loc}. \tag{3.40}$$

Therefore, $\{\widetilde{\beta}_\varepsilon(u_\varepsilon)\}$ is bounded in $L^q_{\rm loc}$ (as a matter of fact, since $|\widetilde{\beta}_\varepsilon(r)| \le C_M(1+|r|)$, $|r| \le M$, $\varepsilon \in (0,1)$, see (3.54) below, it is bounded in L^∞). Hence, also by selecting a further subsequence $\{\varepsilon\} \to 0$, we have

$$\widetilde{\beta}_\varepsilon(u_\varepsilon) \to \eta \text{ weakly in } L^q_{\rm loc}, \tag{3.41}$$

where $q \in \left(1, \frac{d}{d-1}\right)$.

Now, we consider two cases:

Case 1. b is constant (hence $b_\varepsilon \equiv b$).

By (3.21), (3.40), (3.41), we have

$$u - \lambda\Delta\eta + \lambda\ {\rm div}(Dbu) = f \text{ in } \mathcal{D}'(\mathbb{R}^d). \tag{3.42}$$

It remains to be shown that

$$\eta(x) = \beta(u(x)),\ \text{a.e. } x \in \mathbb{R}^d. \tag{3.43}$$

For this purpose, we shall prove first, via the Riesz-Kolmogorov compactness theorem, that $\{\widetilde{\beta}_\varepsilon(u_\varepsilon)\}_{\varepsilon>0}$ is compact in $L^1_{\rm loc}$. We set

$$v_\varepsilon=\widetilde{\beta}_\varepsilon(u_\varepsilon),\ \ v^\nu_\varepsilon(x)=v_\varepsilon(x+\nu)-v_\varepsilon(x),\ \ \forall x,\nu\in\mathbb{R}^d.$$

By (3.37), we have

$$v^\nu_\varepsilon=\frac1\lambda\,E^\nu*(-u_\varepsilon-\varepsilon\lambda\widetilde{\beta}_\varepsilon(u_\varepsilon)+f)-\nabla(E^\nu*(Dbu_\varepsilon)),\ \ E^\nu(x)\equiv E(x+\nu)-E(x),$$

and, by (1.4), (1.5), this yields, for any compact $K\subset\mathbb{R}^d$, $1<q<\frac{d}{d-1}$, $q<p<\frac{d}{d-2}$, and all $\varepsilon\in(0,1)$,

$$\begin{aligned}\|v^\nu_\varepsilon\|_{L^q(K)}&\le\frac{C_K}{\lambda}\,\|E^\nu*(-u_\varepsilon-\varepsilon\lambda\widetilde{\beta}_\varepsilon(u_\varepsilon)+f)\|_{L^p(K)}+\|\nabla(E^\nu*(Dbu_\varepsilon))\|_{L^q(K)}\\&\le\frac{C_K}{\lambda}\,\|E^\nu*(-u_\varepsilon-\varepsilon\lambda\widetilde{\beta}_\varepsilon(u_\varepsilon)+f)\|_{M^{\frac{d}{d-2}}}+C_K\|\nabla(E^\nu*(Dbu_\varepsilon))\|_{M^{\frac{d}{d-1}}}\\&\le\frac{C_K}{\lambda}\,\|E^\nu\|_{M^{\frac{d}{d-2}}}(|u_\varepsilon|_1+|f|_1)+C_K\|\nabla E^\nu\|_{M^{\frac{d}{d-1}}}|u_\varepsilon|_1\\&\le C_K\left(1+\frac1\lambda\right)(\|E^\nu\|_{M^{\frac{d}{d-2}}}+\|\nabla E^\nu\|_{M^{\frac{d}{d-1}}})|f|_1.\end{aligned}$$

On the other hand, we have

$$\lim_{\nu\to0}(\|E^\nu\|_{M^{\frac{d}{d-2}}}+\|\nabla E^\nu\|_{M^{\frac{d}{d-1}}})=0.$$

(This continuity property follows as in the case of L^p-spaces.) Therefore, $\{\widetilde{\beta}_\varepsilon(u_\varepsilon);\ \varepsilon\in(0,1)\}$ is compact in each space $L^q(K)$, K compact subset of $\mathbb{R}^d$, where $1<q<\frac{d}{d-1}$. We also note that $\varepsilon|u_\varepsilon|_1\to0$ as $\varepsilon\to0$. Hence, on a subsequence $\{\varepsilon\}\to0$,

$$\beta_\varepsilon(u_\varepsilon)\to\eta\text{ strongly in }L^q_{\rm loc}. \tag{3.44}$$

Since $\{u_\varepsilon;\ \varepsilon\in(0,1)\}$ are bounded in L^∞, we have

$$\lim_{\varepsilon\to0}|\beta_\varepsilon(u_\varepsilon)-\beta(u_\varepsilon)|_\infty=0,$$

so, by (3.44),

$$\beta(u_\varepsilon)\to\eta\text{ strongly in }L^q_{\rm loc}. \tag{3.45}$$

Recalling that $u_\varepsilon \to u$ weak-star in L^∞ and that the map $u \to \beta(u)$ is maximal monotone in each dual pair $(L^q(K), L^{q'}(K))$, hence weakly-strongly closed, we get by (3.45) that (3.43) holds. Hence, by (3.32), (3.40), $u = u(\lambda, f)$ satisfies (3.18) and we have

$$|u(\lambda,f)-u(\lambda,g)|_1 \le |f-g|_1,\ \forall \lambda>0,\ f,g\in L^1\cap L^\infty. \tag{3.46}$$

(Indeed, first only for $0<\lambda<\lambda_0$, but then by Proposition 3.1 in [2] for all $\lambda>0$.)

Case 2. Let β be strictly increasing and $\operatorname{div} D \ge 0$.

Multiplying (3.31) by $u_\varepsilon = u_\varepsilon(f)$ and integrating over $\mathbb{R}^d$, since $u_\varepsilon, b_\varepsilon(u_\varepsilon)u_\varepsilon \in H^1\cap L^1\cap L^\infty$, $\widetilde{\beta}_\varepsilon(u_\varepsilon)\in H^2$, we obtain

$$\begin{aligned} &|u_\varepsilon|_2^2 + \lambda\int_{\mathbb{R}^d}\beta'_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^2dx\\ &\qquad\le \lambda\int_{\mathbb{R}^d}(\nabla u_\varepsilon\cdot D)b_\varepsilon(u_\varepsilon)u_\varepsilon dx + \frac12\ |u_\varepsilon|_2^2+\frac12\ |f|_2^2. \end{aligned}\tag{3.47}$$

Defining

$$\psi(r)=\int_0^r b_\varepsilon(s)s\,ds,\ r\in\mathbb{R},$$

we see that $\psi\ge 0$, hence the first integral on the right hand side of (3.47) is equal to

$$-\int_{\mathbb{R}}\operatorname{div} D\psi(u_\varepsilon)dx \le 0. \tag{3.48}$$

Define $g_\varepsilon(r)=(I+\varepsilon\beta)^{-1}(r)$, $r\in\mathbb{N}$, and

$$a(r)=\int_0^r \frac{\beta'(s)}{1+\beta'(s)}\ ds,\quad r\in\mathbb{R}.$$

Since

$$\beta'_\varepsilon(r)\ge\frac{\beta'(g_\varepsilon(r))}{1+\beta'(g_\varepsilon(r))}\ge\frac{\beta'(g_\varepsilon(r))}{1+\beta'(g_\varepsilon(r))}\ (g'_\varepsilon(r))^2,\ r\in\mathbb{R},$$

and thus

$$\beta'_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^2 \ge |\nabla a(g_\varepsilon(u_\varepsilon))|^2,$$

we obtain from (3.47), (3.48)

$$|u_\varepsilon|_2^2+2\lambda\int_{\mathbb{R}^d}|\nabla a(g_\varepsilon(u_\varepsilon))|^2dx\le|f|_2^2. \tag{3.49}$$

Since $|a(r)| \leq |r|$ and $|g_\varepsilon(r)| \leq |r|$, $r \in \mathbb{R}$, this implies that $a(g_\varepsilon(u_\varepsilon))$, $\varepsilon > 0$, is bounded in H^1, hence compact in $L^2_{\rm loc}$, so along a subsequence $\varepsilon \to 0$

$$a(g_\varepsilon(u_\varepsilon)) \to v \ \text{ in } L^2_{\rm loc} \text{ and a.e.}$$

Since a is strictly increasing and continuous, and thus so is its inverse function a^{-1}, it follows that

$$g_\varepsilon(u_\varepsilon) \to a^{-1}(v), \ \text{a.e. } ,$$

and so, as $\varepsilon \to 0$,

$$u_\varepsilon = g_\varepsilon(u_\varepsilon) + \varepsilon\beta(g_\varepsilon(u_\varepsilon)) \to a^{-1}(v), \ \text{a.e. on } \mathbb{R}^d.$$

Therefore, by (3.34) we have $u \in L^\infty$ and by (3.40)

$$u_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u \ \text{ in } L^p_{\rm loc}, \quad \forall p \in [1, \infty). \tag{3.50}$$

By Fatou's lemma and (3.32), it follows that $u \in L^1$. Furthermore, obviously both $\widetilde{\beta}_\varepsilon$ and b_ε, $\varepsilon \in (0, 1)$, are locally equicontinuous. Hence,

$$\widetilde{\beta}_\varepsilon(u_\varepsilon) \to \beta(u), \quad b_\varepsilon(u_\varepsilon)u_\varepsilon \to b(u)u, \ \text{a.e. on } \mathbb{R}^d, \tag{3.51}$$

as $\varepsilon \to 0$, since $\sup\limits_{\varepsilon>0} |u_\varepsilon|_\infty < \infty$ by (3.35). We note that, since β is locally Lipschitz and $\beta(0) = 0$, we have, for $M > 0$, $C_M = \sup\limits_{|r|\leq M} \beta'(r)$,

$$|\beta_\varepsilon(r)| \leq C_M |(I + \varepsilon\beta)^{-1} r| \leq C_M |r|, \ r \in [-M, M]. \tag{3.52}$$

Hence, by (3.35), (3.50) and because $|b_\varepsilon| \leq |b|_\infty$, (3.52) implies that both convergences in (3.51) also hold in $L^p_{\rm loc}$, $p \in [1, \infty)$. Hence we can pass to the limit in (3.31) to conclude that u satisfies (3.18), which in turn implies that $u \in D(A)$, because $\beta(u) \in L^p_{\rm loc}$, $p \in [1, \infty)$. Furthermore, Fatou's lemma and (3.32) imply that (3.46) also holds in Case 2.

All what follows now holds in both Cases 1 and 2. We define $J_\lambda : L^1 \cap L^\infty \to L^1$

$$J_\lambda(f) = u(\lambda, f), \quad \forall\, f \in L^1 \cap L^\infty, \ \lambda > 0.$$

Then, (3.4) follows by definition. We note that, by estimate (3.39), it also follows

$$|\beta(u(\lambda, f))|_{L^q(K)} \leq C_K \left(1 + \frac{1}{\lambda}\right) |f|_1, \ \forall f \in L^1 \cap L^\infty, \ \lambda > 0. \tag{3.53}$$

Now, let $f \in L^1$ and $\{f_n\} \subset L^1 \cap L^\infty$ be such that $f_n \to f$ in L^1. If $u_n = u(\lambda, f_n)$, we see by (3.46) that $u_n \to u$ in L^1 and, by (3.53), $\{\beta(u_n)\}$ is bounded in $L^q(K)$ for each $K \subset \mathbb{R}^d$, $q \in \left(0, \frac{d}{d-1}\right)$. Hence, by the generalized Lebesgue convergence theorem and the continuity of β, $\beta(u_n) \to \beta(u)$ in $L^q(K)$, and so $u \in D(A_0)$, $A_0 u_n \to A_0 u$ in L^1, and

$$u + \lambda A_0 u = f, \tag{3.54}$$

Then, we may extend (3.46) and (3.53) to all of $f \in L^1$ and hence (3.3) and (3.5) hold.

By (3.34), (3.40), it follows that (3.6) holds. Moreover,if $f \geq 0$ on $\mathbb{R}^d$, by (3.21) it is easily seen that $u_\varepsilon \geq 0$ on $\mathbb{R}^d$, and so, by (3.40) we infer that $u \geq 0$. Moreover, we have

$$\int_{\mathbb{R}^d} u\, dx = \int_{\mathbb{R}^d} f\, dx. \tag{3.55}$$

By (3.54), we have

$$\begin{aligned}\int_{\mathbb{R}^d} u\varphi dx - \lambda \int_{\mathbb{R}^d} \beta(u)\Delta\varphi dx + \lambda \int_{\mathbb{R}^d} ub(u)D \cdot \nabla\varphi dx \\ = \int_{\mathbb{R}^d} f\varphi\, dx,\ \forall \varphi \in C_0^\infty.\end{aligned} \tag{3.56}$$

Now, we choose in (3.56) $\varphi = \varphi_\nu \in C_0^\infty$, where $\varphi_\nu \to 1$ on $\mathbb{R}^d$, $0 \leq \varphi_\nu \leq 1$, and

$$|\Delta\varphi_\nu|_\infty + |\nabla\varphi_\nu|_\infty \to 0 \text{ as } \nu \to 0.$$

(Such an example is $\varphi_\nu(x) = \exp\left(-\frac{\nu|x|^2}{1-\nu|x|^2}\right)$.)

This implies (3.55), and so (3.8) holds.

It remains to prove (3.9). Let $g \in C_0^\infty(\mathbb{R}^d)$ be arbitrary but fixed. Coming back to equation (3.31), we can write

$$g - \lambda\Delta\widetilde{\beta}_\varepsilon(g) + \lambda\varepsilon\widetilde{\beta}_\varepsilon(g) + \lambda\, \mathrm{div}(Db_\varepsilon(g)g) = g + \lambda A_\varepsilon(g),$$

and so (3.32) yields

$$|u_\varepsilon(g) - g|_1 \leq \lambda|A_\varepsilon(g)|_1 \leq C\lambda\|g\|_{W^{2,2}(\mathbb{R}^d)},$$

because $\beta \in C^2$ and hence $\widetilde{\beta}_\varepsilon$, $(\widetilde{\beta}_\varepsilon)'$ and $(\widetilde{\beta}_\varepsilon)''$ are locally uniformly bounded in $\varepsilon \in (0,1)$ and $|b_\varepsilon| \leq |b|_\infty$. This, together with (3.40), implies (3.9), as claimed.

This completes the proof of Lemma 3.1 for $d \geq 3$.

Now, we shall sketch the proof of the case $d = 1$ only in the case that $b \equiv 1$. By (3.36), we have

$$(\widetilde{\beta}_\varepsilon(u_\varepsilon))'(x) = D(x)u_\varepsilon(x) + \lambda^{-1} \int_{-\infty}^{x} (u_\varepsilon(y) - f(y))dy, \ \forall x \in \mathbb{R},$$

and, therefore,

$$\begin{aligned} \widetilde{\beta}_\varepsilon(u_\varepsilon(x)) &= \widetilde{\beta}_\varepsilon(u_\varepsilon(x_0)) + \int_{x_0}^{x} D(y)u_\varepsilon(y)dy \\ &\quad + \lambda^{-1} \int_{x_0}^{x} ds \int_{-\infty}^{s} (u_\varepsilon(y) - f(y))dy, \ \forall x, x_0 \in \mathbb{R}. \end{aligned} \tag{3.57}$$

Taking into account that $\{u_\varepsilon\}$ is bounded in L^1, we may choose x_0 independent of ε such that $\{u_\varepsilon(x_0)\}$ is bounded. This implies that $\{\widetilde{\beta}_\varepsilon(u_\varepsilon)\}_{\varepsilon>0}$ is bounded in $L^\infty_{\rm loc}$ and so estimate (3.39) follows. Hence, it follows as above that (3.40)-(3.42) follow too. To prove (3.43), we shall prove that $\{\widetilde{\beta}_\varepsilon(u_\varepsilon)\}_{\varepsilon>0}$ is compact in $L^1_{\rm loc}$. If $v_\varepsilon = \beta_\varepsilon(u_\varepsilon)$ and $v^\nu_\varepsilon = v_\varepsilon(x+\nu) - v_\varepsilon(x)$, we get by (3.57) that

$$v^\nu_\varepsilon(x) = \int_{x}^{x+\nu} D(y)u_\varepsilon(y)dy + \lambda^{-1} \int_{x}^{x+\nu} ds \int_{-\infty}^{s} (u_\varepsilon(y) - f(y))dy$$

and, therefore, $\lim\limits_{\nu\to 0} |v^\nu_\varepsilon|_{L^1_{\rm loc}} = 0$ uniformly with respect to ε. Hence, $\{v_\varepsilon\}$ is compact in $L^1_{\rm loc}$ and so (3.43) follows by (3.44). As regard (3.46)-(3.55), and so all conclusion of Lemma 3.1, it follows as in the previous case.

Later, we shall also need the following convergence result for the solution u_ε to the approximating equation (3.21).

Lemma 3.3. *Assume that β is strictly iuncreasing. Then, we have, for $\varepsilon \to 0$,*

$$u_\varepsilon \to u = J_\lambda f \ \text{ in } L^1, \ \forall\, f \in L^1. \tag{3.58}$$

Proof. We shall proceed as in the proof of Lemma 3.2 in [7]. It suffices to prove this for f in a dense subset of L^1. In Case 2 of the proof of Lemma 3.1, by (3.50) we have $u_\varepsilon \to u = J_\lambda f$ strongly in $L^1_{\rm loc}$. But this also follows in Case 1, because by (3.43) and (3.45) we have that $\beta(u_\varepsilon) \to \beta(u)$ in $L^1_{\rm loc}$. So, by our additional assumption that β is strictly increasing, which entails that its inverse β^{-1} is continuous, we have that also $u_\varepsilon \to u$, a.e. (along a subsequence), hence $u_\varepsilon \to u$ in $L^1_{\rm loc}$ by (3.35). Hence it suffices to show that there is C independent of ε such that

$$\|u_\varepsilon\| = \int_{\mathbb{R}^d} |u_\varepsilon(x)|\Phi(x)dx \le C, \quad \forall\, \varepsilon > 0, \tag{3.59}$$

where $\Phi \in C^2(\mathbb{R}^d)$ is such that $1 \le \Phi(x),\ \forall\, x \in \mathbb{R}^d$, and

$$\Phi(x) \to +\infty \ \text{ as } |x| \to \infty,\ \nabla\Phi \in L^\infty,\ \Delta\Phi \in L^\infty.$$

(An example is $\Phi(x) \equiv (1+|x|^2)^\alpha$ with $\alpha \in \left(0, \frac{1}{2}\right]$.)

We fix such a function Φ and assume that

$$f \in L^1 \cap L^\infty,\quad \|f\| = \int_{\mathbb{R}^d} \Phi(x)|f(x)|dx < \infty.$$

If we multiply equation (3.21) by $\varphi_\nu \mathcal{X}_\delta(\widetilde{\beta}_\varepsilon(u_\varepsilon))$, where $\varphi_\nu(x) = \Phi(x)\exp(-\nu\Phi(x))$, $\nu > 0$, and integrate over $\mathbb{R}^d$, we get, since $\mathcal{X}'_\delta \ge 0$,

$$\begin{aligned}
\int_{\mathbb{R}^d} u_\varepsilon \mathcal{X}_\delta(\widetilde{\beta}_\varepsilon(u_\varepsilon))\varphi_\nu\, dx &\le -\lambda \int_{\mathbb{R}^d} \nabla\widetilde{\beta}_\varepsilon(u_\varepsilon)\cdot\nabla(\mathcal{X}_\delta(\widetilde{\beta}_\varepsilon(u_\varepsilon))\varphi_\nu)dx\\
&\quad +\lambda \int_{\mathbb{R}^d} Db^*_\varepsilon(u_\varepsilon)\cdot\nabla(\mathcal{X}_\delta(\widetilde{\beta}_\varepsilon(u_\varepsilon))\varphi_\nu)dx + \int_{\mathbb{R}^d} |f|\varphi_\nu dx\\
&\le -\lambda \int_{\mathbb{R}^d} (\nabla\widetilde{\beta}_\varepsilon(u_\varepsilon)\cdot\nabla\varphi_\nu)\mathcal{X}_\delta(\widetilde{\beta}_\varepsilon(u_\varepsilon))dx\\
&\quad +\lambda \int_{\mathbb{R}^d} Db^*_\varepsilon(u_\varepsilon)\cdot\nabla\widetilde{\beta}_\varepsilon(u_\varepsilon)\mathcal{X}'_\delta(\widetilde{\beta}_\varepsilon(u_\varepsilon))\varphi_\nu dx\\
&\quad +\lambda \int_{\mathbb{R}^d} (D\cdot\nabla\varphi_\nu)b^*_\varepsilon(u_\varepsilon)\mathcal{X}_\delta(\widetilde{\beta}_\varepsilon(u_\varepsilon))dx + \int_{\mathbb{R}^d} |f|\varphi_\nu dx.
\end{aligned} \tag{3.60}$$

Here, $b^*_\varepsilon(u) = b_\varepsilon(u)u$. Letting $\delta \to 0$, we get as above

$$\begin{aligned}
\int_{\mathbb{R}^d} |u_\varepsilon|\varphi_\nu dx &\le -\lambda \int_{\mathbb{R}^d} \nabla|\widetilde{\beta}_\varepsilon(u_\varepsilon)|\cdot\nabla\varphi_\nu dx\\
&\quad +\overline{\lim_{\delta\to 0}}\frac{\lambda}{\delta}\int_{[|\widetilde{\beta}_\varepsilon(u_\varepsilon)|\le\delta]} |D|\,|b^*_\varepsilon(u_\varepsilon)|\,|\nabla\widetilde{\beta}_\varepsilon(u_\varepsilon)|\varphi_\nu dx\\
&\quad +\lambda \int_{\mathbb{R}^d} (\operatorname{sign} u_\varepsilon)b^*_\varepsilon(u_\varepsilon)(D\cdot\nabla\varphi_\nu)dx + \int_{\mathbb{R}^d} |f|\varphi_\nu dx\\
&\le \lambda \int_{\mathbb{R}^d} (|\widetilde{\beta}_\varepsilon(u_\varepsilon)|\Delta\varphi_\nu + |b^*_\varepsilon(u_\varepsilon)|\,|D\cdot\nabla\varphi_\nu|)dx + \int_{\mathbb{R}^d} |f|\varphi_\nu dx,
\end{aligned} \tag{3.61}$$

where in the last step we used that

$$|b^*_\varepsilon(u_\varepsilon)| \le \mathrm{Lip}(b^*_\varepsilon)|u_\varepsilon| \le \frac{1}{\varepsilon}\,\mathrm{Lip}(b^*_\varepsilon)|\widetilde{\beta}_\varepsilon(u_\varepsilon)|.$$

We have

$$\begin{aligned}
&\nabla\varphi_\nu(x) = (\nabla\Phi - \nu\Phi\nabla\Phi)\exp(-\nu\Phi),\\
&\Delta\varphi_\nu(x) = (\Delta\Phi - \nu|\nabla\Phi|^2 - \nu\Phi\Delta\Phi + \nu^2\Phi|\nabla\Phi|^2 - \nu|\nabla\Phi|^2)\exp(-\nu\Phi).
\end{aligned}$$

Then, letting $\nu \to 0$ in (3.61), since $M := \sup\limits_{\varepsilon>0}|u_\varepsilon|_\infty < \infty$, $|b^*_\varepsilon(r)| \le |b|_\infty|r|$, and $|\widetilde{\beta}_\varepsilon(r)| \le \left(\sup\limits_{|r|\le M}\beta'(r)+\varepsilon\right)|r|$, $\forall r \in [-M,M]$, we get

$$\|u_\varepsilon\| \le \|f\| + C\lambda(|\Delta\Phi|_\infty + |D|_\infty|\nabla\Phi|_\infty)|f|_1,\ \forall \varepsilon \in (0,1).$$

Hence (3.59) and, therefore, (3.58), holds for all $f \in L^1 \cap L^\infty$ with $\|f\| < \infty$. Since the latter set is dense in L^1, we get (3.58), as claimed.

Proof of Theorem 2.2 (continued). As seen earlier, the solution u_h to the finite difference scheme (2.2)–(2.4) is uniformly convergent on every compact interval $[0,T]$ to $u \in C([0,\infty);L^1)$. By (3.6) and (3.17), by a standard argument we obtain that, for $u_0 \in L^1 \cap L^\infty$,

$$|u(t)|_\infty = |S(t)u_0|_\infty \le \exp(|(\mathrm{div}\,D)^-|_\infty^{\frac12}t)|u_0|_\infty,\ \forall t \ge 0.$$

(2.7) follows by (3.17) and (3.8).

Let us prove now that u is a distributional solution to (1.1). We note first that by (2.4) we have (setting $u_h(t) = u_0$ for $t \in (-\infty,0)$)

$$\begin{aligned}
&u_h(t) - h\Delta\beta(u_h(t)) + h\ \mathrm{div}(Du_h(t)) = u_h(t-h),\ t \ge 0,\\
&u_h(0) = u_0.
\end{aligned} \tag{3.62}$$

Since $\lim\limits_{h\to0} u_h(t) = S(t)u_0$ in L^1 locally uniformly in $t \in [0,\infty)$, we have by (2.6) that

$$|u_h(t)|_\infty \le \exp(|(\mathrm{div}\,D)^-|_\infty^{\frac12}t)|u_0|_\infty + 1,\ t \ge 0,$$

for small enough h, and, hence, for $h \to 0$, $\beta(u_h(t)) \to \beta(u(t))$ in L^1_{loc}, a.e. $t > 0$.

Let $\varphi \in C_0^\infty([0,\infty) \times \mathbb{R}^d)$. Then, by (3.62) we have

$$\int_0^\infty \int_{\mathbb{R}^d} \frac{1}{h} \, (u_h(t,x) - u_h(t-h,x))\varphi(t,x) - \beta(u_h(t,x))\Delta\varphi(t,x)$$
$$-b(u_h(t,x))u_h(t,x)D(x) \cdot \nabla\varphi(t,x)dt\,dx = 0,$$

if take $u_h(t,x) \equiv u_0(x)$ for $t \in (-h, 0]$.

Then, replacing the first term by

$$\int_0^\infty \int_{\mathbb{R}^N} \frac{1}{h} \, u_h(t,x)(\varphi(t+h,x) - \varphi(t,x))dt\,dx + \frac{1}{h} \int_0^h \int_{\mathbb{R}^N} u_0(x)\varphi(t,x)dt\,dx$$

and, letting $h \to 0$, we get (2.9), as claimed. This completes the proof. $\square$

Remark 3.4. If $u_0 \geq 0$, then the generalized solution u given by Theorem 2.2 is nonnegative and $\beta(u) \in L^\infty(0,T;L^1)$ for all $T > 0$. Indeed, by (2.2)-(2.4) we see by estimate (3.39) that

$$h\|\beta(u_h^{i+1})\|_{L^q(K)} \leq C_K^1(|u_h^{i+1} - u_h^i|_1 + h|u_h^{i+1}|_1), \; i = 0,1,...,N-1.$$

Taking into account that $\beta(u_h^{i+1}) \geq 0$, $\forall i = 0,1,...,N-1$, we obtain the estimate

$$\int_0^T \beta(u_h(t,x))dx \leq C(|u_h(T)|_1 + |u_0|_1), \tag{3.63}$$

which implies the desired result. By the previous argument, this implies that, for $u_0 \geq 0$, $u_0 \in L^1$, the solution u is a Schwartz distribution solution to (1.1).

Remark 3.5. We note that, if β is nondegenerate, that is, $\beta'(r) \geq a > 0$, $\forall r \in \mathbb{R}$, then it follows that $(I + \lambda A_0)^{-1}L^1 = D(A_0)$ (see [7]) and this implies that $D(A_0) = D(A)$ and so $A = A_0$.

Remark 3.6. Theorem 2.2 extends by a slight modification of the proof to the multivalued functions $\beta : \mathbb{R} \to 2^\mathbb{R}$ with $R(B) = (-\infty, +\infty)$ and which are maximal monotone graphs on $\mathbb{R} \times \mathbb{R}$, that is,

$$(v_1 - v_2)(u_1 - u_2) \geq 0, \; \forall v_i \in \beta(u_i), \; i = 1,2,$$

and $R(1 + \beta) = \mathbb{R}$. We omit the proof details, but we note, however, that this case covers the case of FPE (1.1) with monotonically nondecreasing discontinuous functions β and, in particular, the self-organized criticality model (see, e.g., [5])

$$\beta(u) \equiv \alpha H(u - u_c)u,$$

where $\alpha > 0$, H is the Heviside function and $u_c > 0$. As a matter of facts, if β_0 is a monotonically nondecreasing function with a numerable set of jumps $\{r_j\}_{j=1}^{\infty}$, it has an extension to a maximal monotone graph β by filling the jumps, that is, setting

$$\beta(r_j) = [\beta_0(r_j + 0), \beta(r_{j+1} - 0)],\ j = 1, ...$$

4 Regularizing effect on initial data

Consider here equation (1.1) under the following hypotheses.

(k) $\beta \in C^2(\mathbb{R}),\ \beta'(r) \ge a|r|^{\alpha-1},\ \forall r \in \mathbb{R};\ \beta(0) = 0,$
where $\alpha \ge 1$, $d \ge 3$, $a > 0$.

(kk) $D \in (L^2 \cap L^\infty)(\mathbb{R}^d; \mathbb{R}^d)$, div $D \in L^\infty(\mathbb{R}^d)$, div $D \ge 0$, a.e.

(kkk) $b \in C_b(\mathbb{R}) \cap C^1(\mathbb{R}),\ b \ge 0.$

We have

Theorem 4.1. *Let $d \ge 3$. Then, under Hypotheses* (k), (kk), (kkk), *the generalized solution u to* (1.1) *given by Theorem* 2.2 *for $\mu = u_0dx$, $u_0 \in L^1$, satisfies*

$$u(t) \in L^\infty,\ \forall t > 0, \tag{4.1}$$

$$|u(t)|_\infty \le C\, t^{-\frac{d}{2+(\alpha-1)d}}\ |u_0|_1^{\frac{2}{2+d(\alpha-1)}},\ \forall t \in (0, \infty),\ u_0 \in L^1, \tag{4.2}$$

where C is independent of u_0.

Proof. We shall first prove the following lemma.

Lemma 4.2. *Let $u_\lambda = (I + \lambda A)^{-1} f$, where A is the operator* (3.10), (3.11). *Then, for each $p > 1$ and $\lambda > 0$, we have*

$$|u_\lambda|_p^p + \lambda a C \frac{p(p-1)}{(p+\alpha-1)^2}\, |u_\lambda|^{p+\alpha-1}_{\frac{(p+\alpha-1)d}{d-2}} \le |f|_p^p,\ \forall f \in L^p \cap L^1. \tag{4.3}$$

where C is independent of p and λ.

Proof. Let $p \in (1,\infty)$. By approximation, we may assume that $f \in L^1 \cap L^\infty$. Let us first explain the proof by heuristic computations in the case $b \equiv 1$, that is, for the equation

$$u_\lambda - \lambda\Delta\beta(u_\lambda) + \lambda\,\mathrm{div}(Du_\lambda) = f. \tag{4.4}$$

We multiply (3.21) by $|u_\lambda|^{p-2}u_\lambda$ and integrate over $\mathbb{R}^d$ and get

$$\begin{aligned} |u_\lambda|_p^p + (p-1)\lambda \int_{\mathbb{R}^d} \beta'(u_\lambda)|\nabla u_\lambda|^2 |u_\lambda|^{p-2} dx \\ = \int_{\mathbb{R}^d} f|u_\lambda|^{p-2}u_\lambda dx - \frac{(p-1)}{p}\,\lambda \int_{\mathbb{R}^d} |u_\lambda|^p \mathrm{div}\, D\,dx \\ \le |f|_p |u_\lambda|_p^{p-1} \le \frac{1}{p}\,|f|_p^p + \left(1-\frac{1}{p}\right)|u_\lambda|_p^p. \end{aligned} \tag{4.5}$$

Taking into account (k) yields

$$|u_\lambda|_p^p + ap(p-1)\lambda \int_{\mathbb{R}^d} |u_\lambda|^{p+\alpha-3}|\nabla u_\lambda|^2 dx \le |f|_p^p.$$

On the other hand,

$$\begin{aligned} \int_{\mathbb{R}^d} |u_\lambda|^{p+\alpha-3}|\nabla u_\lambda|^2 dx &= \left(\frac{2}{p+\alpha-1}\right)^2 \int_{\mathbb{R}^d} \left|\nabla\left(|u_\lambda|^{\frac{p+\alpha-1}{2}}\right)\right|^2 dx \\ &\ge C\left(\frac{2}{p+\alpha-1}\right)^2 \left(\int_{\mathbb{R}^d} |u_\lambda|^{\frac{(\alpha-1+p)d}{d-2}}\,dx\right)^{\frac{d-2}{d}} \end{aligned}$$

(by the Sobolev embedding theorem in $\mathbb{R}^d$). This yields

$$|u_\lambda|_p^p + \lambda aC\,\frac{p(p-1)}{(p+\alpha-1)^2}\,|u_\lambda|^{\alpha-1+p}_{\frac{(\alpha-1+p)d}{d-2}} \le |f|_p^p,\ \forall\lambda > 0,$$

as claimed.

Of course, the above argument is heuristic. Since e.g. $u_\lambda \notin H^1$. To make the proof rigorous, we recall that by Lemma 3.3 the solution $u := u_\lambda$ to (4.4) constructed in Lemma 3.2 is an L^1-limit of solutions u_ε, $\varepsilon > 0$, to the approximating equations (3.31) (with A_ε as in (3.30)). So, we shall start with (3.31) (instead of (4.4)) and with its solution u_ε. Then we know by the proof of Lemma 3.1 (see (3.23) and (3.35)) that $u_\varepsilon, b(u_\varepsilon)u_\varepsilon \in H^1 \cap L^1 \cap L^\infty$, $\widetilde{\beta}_\varepsilon(u_\varepsilon) \in H^2$. We have, for all $r \in \mathbb{R}$,

$$\widetilde{\beta}'_\varepsilon(r) = \frac{\beta'(g_\varepsilon(r))}{(1+\varepsilon\beta')(g_\varepsilon(r))} + \varepsilon \geq h_\varepsilon(g_\varepsilon(r)), \tag{4.6}$$

where

$$h_\varepsilon(r) = \frac{a|r|^{\alpha-1}}{1+\varepsilon a|r|^{\alpha-1}}, \quad r \in \mathbb{R}, \tag{4.7}$$

and $g_\varepsilon = (I+\varepsilon\beta)^{-1}$.

Define $\varphi_\delta : \mathbb{R} \to \mathbb{R}$ by

$$\varphi_\delta(r) = (|r|+\delta)^{p-2}r, \quad r \in \mathbb{R}.$$

Then, $\varphi_\delta \in C_b^1$, $\lim\limits_{\delta\to 0} \varphi'_\delta(r) = (p-1)|r|^{p-2}$ and $\varphi'_\delta(r) \geq \min(1, p-1)(|r|+\delta)^{p-2}$. Now, we multiply (3.31) by $\varphi_\delta(u_\varepsilon)$ and obtain

$$\begin{aligned} &\int_{\mathbb{R}^d} u_\varepsilon \varphi_\delta(u_\varepsilon)dx + \lambda \int_{\mathbb{R}^d} \widetilde{\beta}'_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^2 \varphi'_\delta(u_\varepsilon)dx \\ &\qquad = \lambda \int_{\mathbb{R}^d} (D\cdot\nabla u_\varepsilon)\varphi'_\delta(u_\varepsilon)b_\varepsilon(u_\varepsilon)u_\varepsilon dx + \int_{\mathbb{R}^d} f\varphi_\delta(u_\varepsilon)dx. \end{aligned} \tag{4.8}$$

Defining

$$\psi(r) = \int_0^r \varphi'_\delta(s)b_\varepsilon(s)s\,ds,$$

we see that $\psi \geq 0$, hence the first integral in the right hand side of (4.8) is equal to

$$-\int_{\mathbb{R}^d} (\operatorname{div} D)\psi(u_\varepsilon)dx \leq 0. \tag{4.9}$$

Furthermore, we deduce from (4.6) that the second integral on the left hand side of (4.8) dominates

$$\begin{aligned} \int_{\mathbb{R}^d} h_\varepsilon(g_\varepsilon(u_\varepsilon))|\nabla u_\varepsilon|^2 \varphi'_\delta(u_\varepsilon)dx &= \int_{\mathbb{R}^d} |\nabla \psi_{\varepsilon,\delta}(u_\varepsilon)|^2 dx \\ &\geq C\left(\int_{\mathbb{R}^d} |\psi_{\varepsilon,\delta}(u_\varepsilon)|^{\frac{2d}{d-2}}dx\right)^{\frac{d-2}{d}}, \end{aligned} \tag{4.10}$$

where

$$\psi_{\varepsilon,\delta}(r) = \int_0^r \sqrt{h_\varepsilon(g_\varepsilon(s))\varphi'_\delta(s)}\;ds, \quad r \in \mathbb{R}, \tag{4.11}$$

and we used the Sobolev embedding. Combining (4.8)–(4.10) and letting $\delta \to 0$, we obtain by Fatou's lemma and (3.34)

$$|u_\varepsilon|_p^p + \frac{\lambda C}{p} \left(\int_{\mathbb{R}^d} |\psi_\varepsilon(u_\varepsilon)|^{\frac{2d}{d-2}} dx \right)^{\frac{d-2}{2}} \leq \frac{1}{p}\ |f|_p^p + \left(1 - \frac{1}{p}\right) |u_\varepsilon|_p^p, \qquad (4.12)$$

where

$$\psi_\varepsilon(r) = \sqrt{p-1} \int_0^r \sqrt{h_\varepsilon(g_\varepsilon(s))|s|^{p-2}}\ ds, \quad r \in \mathbb{R}. \qquad (4.13)$$

Obviously, $\psi_\varepsilon(u_\varepsilon),\ \varepsilon > 0$, are equicontinuous, hence by (3.34) and (3.50)

$$(\psi_\varepsilon(u_\varepsilon))^2 \to a(p-1) \left(\frac{2}{p+\alpha-1} \right)^2 |u|^{p+\alpha-1}, \quad \text{a.e. on } \mathbb{R}^d.$$

Therefore, by Fatou's lemma, (4.12) implies (4.3).

Proof of Theorem 4.1. We choose $p = p_n$, where $\{p_n\}$ are defined by

$$p_{n+1} = \frac{d}{d-2}\ (p_n + \alpha - 1),\ \ p_0 > 1.$$

Then, by (4.3), we get

$$|u_\lambda|_{p_n}^{p_n} + Ca \frac{p_n(p_n-1)}{(p_n+\alpha-1)^2}\, \lambda |u_\lambda|_{p_{n+1}}^{p_n+\alpha-1} \leq |f|_{p_n}^{p_n},\ \ n = 0, 1, ...$$

Now, we apply Theorem 5.2 in A. Pazy [19], where $\varphi_n(u) = |u|_{p_n}^{p_n}$, $\beta_n = \frac{d-2}{d}$, $C_n = Ca \inf\limits_{p \in [p_0, \infty)} \frac{p(p-1)}{(p+\alpha-1)^2}$, and conclude that (see Proposition 6.5 in [19] or [22]) that

$$|u(t)|_\infty \leq C_{p_0} t^{-\frac{d}{2p_0+(\alpha-1)d}}\ |u_0|_{p_0}^{\frac{2p_0}{2p_0+d(\alpha-1)}},\ \ \forall t > 0,\ \ u_0 \in L^{p_0}. \qquad (4.14)$$

Define

$$C_{\alpha,d} := \frac{d+2}{2d} + \sqrt{(\alpha-1)\left(\alpha + \frac{2}{d}\right) + \left(\frac{d+2}{2d}\right)^2}. \qquad (4.15)$$

Note that, since $\alpha > 1 - \frac{2}{d}$, the value under the root is strictly bigger than $\left(\frac{d-2}{2d}\right)^2$, hence $C_{\alpha,d} > 1$.

Lemma 4.3. *Let $p_0 \in (1, C_{\alpha,d})$. Then, for some constant $C_{p_0} > 0$,*

$$|u(t)|_{p_0} \le C_{p_0}\, t^{-\frac{p_0-\gamma}{p_0(\gamma+\alpha-1)}} |u_0|_1^{\frac{\gamma(p_0+\alpha-1}{p_0(\gamma+\alpha-1)}},\ \forall t > 0,\ u_0 \in L^1 \cap L^{p_0}, \tag{4.16}$$

where

$$\gamma = \frac{2p_0 + (\alpha-1)d}{(p_0+\alpha-2)d+2} \in (0,1). \tag{4.17}$$

Proof. We may assume that $u_0 \in L^\infty$. We shall use the approximating scheme (3.15)-(3.16). By estimate (4.3), we have, for $\lambda = h$,

$$|u_h^{i+1}|_{p_0}^{p_0} + Ch|u_h^{i+1}|^{p_0+\alpha-1}_{\frac{(p_0+\alpha-1)d}{d-2}} \le |u_h^i|^{p_0}_{p_0},\ i = 0, 1, \dots$$

By the summation by parts formula, this yields, for all $t > 0$,

$$t|u_h(t)|^{p_0}_{p_0} + C\int_0^t s|u_h(s)|^{p_0+\alpha-1}_{\frac{(p_0+\alpha-1)d}{d-2}}\, ds \le \int_0^t |u_h(s)|^{p_0}_{p_0} ds + h|u_0|^{p_0}_{p_0}, \tag{4.18}$$

where u_h is given by (3.16) and where, here and below, by C we denote various constants independent of t and u_0, but depending on p_0.

On the other hand, by Hölder's inequality we have

$$|u_h(s)|^{p_0}_{p_0} \le |u_h(s)|_1^\gamma |u_h(s)|^{p_0-\gamma}_{\frac{(p_0+\alpha-1)d}{d-2}},\ s > 0.$$

Then, substituting into (4.18), we get

$$\begin{aligned}
&t|u_h(t)|^{p_0}_{p_0} + C\int_0^t s|u_h(s)|^{p_0+\alpha-1}_{\frac{(p_0+\alpha-1)d}{d-2}}\, ds - h|u_0|^{p_0}_{p_0}\\
&\quad\le |u_0|_1^\gamma \int_0^t |u_h(s)|^{p_0-\gamma}_{\frac{(p_0+\alpha-1)d}{d-2}}\, ds\\
&\quad\le |u_0|_1^\gamma \int_0^t s^{\frac{p_0-\gamma}{p_0+\alpha-1}} |u_h(s)|_{\frac{(p_0+\alpha-1)d}{d-2}})^{p_0-\gamma}\, s^{\frac{\gamma-p_0}{p_0+\alpha-1}} ds\\
&\quad\le |u_0|_1^\gamma \left(\int_0^t s|u_h(s)|^{p_0+\alpha-1}_{\frac{(p_0+\alpha-1)d}{d-2}}\, ds\right)^{\frac{p_0-\gamma}{p_0+\alpha-1}} \left(\int_0^t s^{\frac{\gamma-p_0}{\gamma+\alpha-1}}\, ds\right)^{\frac{\gamma+\alpha-1}{p_0+\alpha-1}}.
\end{aligned}$$

Since $p_0 < C_{\alpha,d}$, we know by Lemma A.1 in the Appendix that

$$\frac{\gamma - p_0}{\gamma+\alpha-1} > -1.$$

Hence the above is dominated by

$$C|u_0|_1^{\gamma}\, t^{\frac{2\gamma+\alpha-p_0-1}{p_0+\alpha-1}} \left(\int_0^t s|u_h(s)|^{p_0+\alpha-1}_{\frac{(p_0+\alpha-1)d}{d-2}}\, ds\right)^{\frac{p_0-\gamma}{p_0+\alpha-1}}.$$

This yields, for $t>0$,

$$t|u_h(t)|_{p_0}^{p_0} + C\int_0^t s|u_h(s)|^{p_0+\alpha-1}_{\frac{(p_0+\alpha-1)d}{d-2}} ds \le C|u_0|_1^{\frac{\gamma(p_0+\alpha-1)}{\gamma+\alpha-1}}\, t^{\frac{2\gamma+\alpha-p_0-1}{\gamma+\alpha-1}} + h|u_0|_{p_0}^{p_0}.$$

Hence, dropping the integral on the left hand side and letting $h\to 0$, we obtain that

$$|u(t)|_{p_0} \le C_{p_0}|u_0|_1^{\frac{\gamma(p_0+\alpha-1)}{p_0(\gamma+\alpha-1)}}\, t^{\frac{\gamma-p_0}{p_0(\gamma+\alpha-1)}},\ t>0,$$

and (4.16) is proved.

Proof of Theorem 4.1 (continued). By approximation, we may assume that $u_0\in L^1\cap L^\infty$. Combining estimates (4.14) and (4.16), we get, for $t>0$,

$$\begin{aligned}
|u(t)|_\infty &\le C_{p_0}\left(\tfrac{t}{2}\right)^{-\frac{d}{2p_0+(\alpha-1)d}}\ \left|u\left(\tfrac{t}{2}\right)\right|_{p_0}^{\frac{2p_0}{2p_0+d(\alpha-1)}}\\
&\le C_{p_0}\ t^{-\left(\frac{d}{2p_0+(\alpha-1)d}+\frac{(p_0-\gamma)2p_0}{p_0(\gamma+\alpha-1)(2p_0+d(\alpha-1))}\right)}\ |u_0|_1^{\frac{\gamma(p_0+\alpha-1)2p_0}{p_0(\gamma+\alpha-1)(2p_0+d(\alpha-1))}} \qquad (4.19)\\
&= C_{p_0}\ t^{-\frac{d}{2+(\alpha-1)d}}\ |u_0|_1^{\frac{2}{2+(\alpha-1)d}},
\end{aligned}$$

where we used Lemma A.2 in the Appendix in the last step and where again the constant C_{p_0} changes from line to line. Hence, Theorem 4.1 is proved.

5 Equation (1.1) with a measure as initial datum

Consider here equation (1.1) with the initial data $u_0=\mu\in\mathcal{M}_b$.

Definition 5.1. The function $u:[0,\infty)\to\mathcal{M}_b$ is a distributional solution to (1.1) if

$$u,\beta(u)\in L^1_{\rm loc}([0,\infty)\times\mathbb{R}^d), \qquad (5.1)$$

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}^d} u(t,x)(\varphi_t(t,x)+b(u(t,x))D(x)\cdot\nabla\varphi(t,x))\\
&\quad+\beta(u(t,x)\Delta\varphi(t,x))dt\,dx+\mu(\varphi(0,\cdot))=0,\forall\varphi\in C_0^\infty([0,\infty)\times\mathbb{R}^d).
\end{aligned} \qquad (5.2)$$

We have

Theorem 5.2. *Assume that Hypotheses* (k), (kk), (kkk) *from Section* 4 *hold and, in addition,*

$$|\beta(r)| \le C|r|^{\alpha}, \quad \forall r \in \mathbb{R}. \tag{5.3}$$

Let $\mu \in \mathcal{M}_b$*. Then,* (1.1) *has a distributional solution which satisfies, for* dt*-a.e.* $t \in (0,\infty)$*,*

$$u(t,x) \ge 0, \text{ a.e. on } \mathbb{R}^d, \text{ provided } \mu \ge 0, \tag{5.4}$$

$$\int_{\mathbb{R}^d} u(t,x)dx = \int_{\mathbb{R}^d} d\mu, \tag{5.5}$$

$$|u(t)|_\infty \le C\, t^{-\frac{d}{2+(\alpha-1)d}} \ \|\mu\|_{\mathcal{M}_b}^{\frac{2}{2+d(\alpha-1)}}, \tag{5.6}$$

$$|u(t)|_1 \le \|\mu\|_{\mathcal{M}_b}. \tag{5.7}$$

Furthermore, for all $p \in \left[1, \alpha + \frac{2}{d}\right)$,

$$u \in L^p((0,T) \times \mathbb{R}^d), \ \forall T > 0, \tag{5.8}$$

and

$$\beta(u) \in L^1((0,T) \times \mathbb{R}^d), \ \forall T > 0. \tag{5.9}$$

The map $t \mapsto u(t,x)dx \in \mathcal{M}_b$ *has a* $\sigma(\mathcal{M}_b, C_b)$*-continuous version on* $(0,\infty)$*, denoted by* $S(t)\mu$*,* $t > 0$*, for which* (5.4)*,* (5.5)*,* (5.6) *and* (5.7) *hold for all* $t > 0$*. Furthermore,*

$$\lim_{t \to 0} \int_{\mathbb{R}^d} (S(t)\mu)(x)\psi(x)dx = \mu(\psi), \ \forall \psi \in C_b. \tag{5.10}$$

Defining $S(0)\mu = \mu$*, then* $S(t)$*,* $t \ge 0$*, restricted to* L^1 *coincides with the semigroup from Theorem 2.2 and we have*

$$\|S(t)\mu_1 - S(t)\mu_2\|_{\mathcal{M}_b} \le \|\mu_1 - \mu_2\|_{\mathcal{M}_b}, \quad \forall\, t \ge 0, \ \mu_1, \mu_2 \in \mathcal{M}_b.$$

Proof. Consider a smooth approximation μ_ε of $u_0 = \mu$ of the form

$$\mu_\varepsilon(x) = (\mu * \rho_\varepsilon), \ \varepsilon > 0,$$

where $\rho_\varepsilon(x) = \frac{1}{\varepsilon}\ \rho\left(\frac{1}{\varepsilon}\,|x|\right)$, $\rho \in C_0^\infty([-1,1])$, $\int_{-1}^1 \rho(r)\,dr = 1$. Then, by Theorem 4.1, the equation

$$\begin{aligned} &(u_\varepsilon)_t - \Delta\beta(u_\varepsilon) + \operatorname{div}(Du_\varepsilon) = 0 \text{ in } (0,\infty) \times \mathbb{R}^d, \\ &u_\varepsilon(0) = \mu_\varepsilon, \end{aligned} \tag{5.11}$$

has, for each $\varepsilon > 0$, a unique generalized solution $u_\varepsilon \in C([0,\infty);L^1) \cap L^\infty((\delta,\infty)\times\mathbb{R}^d)$, $\forall \delta > 0$. More precisely, we have

$$|u_\varepsilon(t)|_\infty \le C\,t^{-\frac{d}{2+(\alpha-1)d}}\;|\mu_\varepsilon|_1^{\frac{2}{2+d(\alpha-1)}} \le C\,t^{-\frac{d}{2+(\alpha-1)d}}\;\|\mu\|_{\mathcal{M}_b}^{\frac{2}{2+d(\alpha-1)}},\ t>0. \tag{5.12}$$

Everywhere in the following, C is a positive constant independent of t and μ possibly changing from line to line. Also, for simplicity, we set $\|\mu\| = \|\mu\|_{\mathcal{M}_b}$ and

$$\|\mu\|^{\frac{2}{2+(\alpha-1)d}}\,t^{-\frac{d}{2+(\alpha-1)d}} = \nu(t,\mu),\ \forall t>0,\ \mu\in\mathcal{M}_b.$$

We also have by (2.8)

$$|u_\varepsilon(t)|_1 \le |\mu_\varepsilon|_1 \le \|\mu\|,\ \forall t \ge 0,\ \forall \varepsilon > 0. \tag{5.13}$$

If we formally multiply (5.11) by $\beta(u_\varepsilon)$ and integrate over $(\delta,t)\times\mathbb{R}^d$, for $\psi(r) = \int_0^r \beta'(s)b(s)s\,ds$, $r\in\mathbb{R}$, since $\psi, \operatorname{div} D \ge 0$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^d} g(u_\varepsilon(t,x))dx + \int_\delta^t |\nabla\beta(u_\varepsilon(s))|_2^2 dx\\
&\qquad= \int_\delta^t \int_{\mathbb{R}^d} \nabla(\psi(u_\varepsilon))\cdot D\,dx\,ds + \int_{\mathbb{R}^d} g(u_\varepsilon(\delta,x))dx\\
&\qquad\le \int_{\mathbb{R}^d} g(u_\varepsilon(\delta,x))dx \le C\|\mu\|(\nu(\delta,\mu))^\alpha,\ \forall t>\delta,
\end{aligned} \tag{5.14}$$

where $g(r) \equiv \int_0^r \beta(s)ds \ge 0$.

Estimate (5.14) can be derived rigorously by using the finite difference scheme (3.15)–(3.16) corresponding to the resolvent of the regularized version (3.31) of equation (5.11). Indeed, by Lemma 3.3, it follows via the Trotter–Kato theorem for nonlinear semigroups (see, e.g., [2], p. 168) that, for each $\varepsilon > 0$,

$$u_\varepsilon(t) = \lim_{\nu\to 0}\lim_{n\to\infty}\left(I + \frac{t}{n}A_\nu\right)^{-n}\mu_\varepsilon,$$

where A_ν is the operator defined by (3.30) and both limits are in L^1, locally uniformly in $t\in[0,\infty)$. Hence,

$$u_\varepsilon(t) = \lim_{\nu\to0}\lim_{h\to0} u_{\nu,h}(t),\ t\in[0,T], \tag{5.15}$$

where

$$\begin{aligned}
&u_{\nu,h}(t) = u_{\nu,h}^{i+1},\ t\in(ih,(i+1)h],\\
&u_{\nu,h}^{i+1} + hA_\nu u_{\nu,h}^{i+1} = u_{\nu,h}^i,\ i=0,1,\dots,N-1;\quad Nh=T,\ u_{\nu,h}^0 = \mu_\varepsilon.
\end{aligned} \tag{5.16}$$

We know by the proof of Lemma 3.1 that, if $v \in L^1 \cap L^\infty$, then for the solution u_h to the equation $u_h + hA_\nu u_h = v$ (see (3.23) and (3.35)), we have $u_h, b_\nu(u_h)u_h \in H^1 \cap L^1 \cap L^\infty$, $\widetilde{\beta}_\nu(u_h) \in H^2$ and $|u_h|_\infty \leq |v|_\infty$. Hence, if we multiply (5.16) by $\widetilde{\beta}_\nu(u^{i+1}_{\nu,h})$ and integrate over $\mathbb{R}^d$, we get as above

$$\begin{aligned}\int_{\mathbb{R}^d} g_\nu(u^{i+1}_{\nu,h}(x))dx + h\int_{\mathbb{R}^d} |\nabla\widetilde{\beta}_\nu(u^{i+1}_{\nu,h})|^2 dx \\ \leq \int_{\mathbb{R}^d} g_\nu(u^i_{\nu,h})dx,\ i = 0,1,...,N_1,\ Nh = T,\end{aligned}$$

where $g_\nu(r) = \int_0^r \widetilde{\beta}_\nu(s)ds$. Summing over from $j = [N\delta/T]+1$ to $k-1 = [Nt/T]$, we get

$$\int_{\mathbb{R}^d} g_\nu(u^k_{\nu,h})dx + \frac{h}{2}\sum_{i=j}^{k-1}\int_{\mathbb{R}^d} |\nabla\widetilde{\beta}_\nu(u^{i+1}_{\nu,h})|^2 dx \leq \int_{\mathbb{R}^d} g_\nu(u^j_{\nu,h})dx,\ \forall k.$$

Then, letting $h \to 0$ and afterwards $\nu \to 0$, by (5.15) and, since $|u^i_{\nu,h}|_\infty \leq |\mu_\varepsilon|_\infty$, the closedness of the gradient on $L^2(0,T;L^2)$ and the weak lower semicontinuity implies (5.14), as claimed.

Multiplying (5.11) by $|u_\varepsilon|^{q-2}u_\varepsilon$, $q \geq 2$, and integrating over $(\delta,t)\times\mathbb{R}^d$, we get by (k)

$$\begin{aligned}(q-1)\int_\delta^t \int_{\mathbb{R}^d} |\nabla |u_\varepsilon|^{\frac{q+\alpha-1}{2}}|^2 ds\,dx + \frac{1}{q}\int_{\mathbb{R}^d} |u_\varepsilon(t,x)|^q dx \\ \leq \frac{1}{q}\int_{\mathbb{R}^d} |u_\varepsilon(\delta,x)|^q dx \leq C\|\mu\|(\nu(\delta,\mu))^{q-1}.\end{aligned} \tag{5.17}$$

As in the previous case, the above calculus can be made rigorous if we replace (5.11) by its discrete version (5.16), which we multiply by $|u^{i+1}_{\nu,h}|^{q-2}u^{i+1}_{\nu,h}$ and integrate over $\mathbb{R}^d$. Indeed, noting that, since $u^{i+1}_{\nu,h}, b_\nu(u^{i+1}_{\nu,h})u^{i+1}_{\nu,h} \in H^1\cap L^1\cap L^\infty$ and $\widetilde{\beta}'_\nu(u^{i+1}_{\nu,h}) \in H^2$ (see (3.23), (3.35)), we have as in (4.9), (4.10) for $\psi_\nu(r) = \sqrt{p-1}\int_0^r \sqrt{h_\nu(g_\nu(s))|s|^{q-2}}\,ds,\ r \in \mathbb{R}$, where $h_\varepsilon, g_\varepsilon$ are as in (4.7).

$$\begin{aligned}\frac{1}{q}\int_{\mathbb{R}^d} |u^{i+1}_{\nu,h}|^q dx + h\int_{\mathbb{R}^d} |\nabla\psi_\nu(u^{i+1}_{\nu,h})|^2 dx \leq \frac{1}{q}\int_{\mathbb{R}^d} |u^i_{\nu,h}|^q dx, \\ i = 0,...,N-1,\ Nh = T.\end{aligned}$$

Summing over i from $j = [N\delta/T] + 1$ to $k - 1 = [Nt/T]$, we get

$$\frac{1}{q}\int_{\mathbb{R}^d} |u^k_{\nu,h}|^q dx + h\sum_{i=j}^{k-1}\int_{\mathbb{R}^d} |\nabla\psi_\nu(u^{i+1}_{\nu,h})|^2 dx \le \frac{1}{q}\int_{\mathbb{R}^d} |u^j_{\nu,h}|^q dx,$$
$$i = 0, ..., N-1,\ Nh = T.$$

Letting $h \to 0$, and afterwards $\nu \to 0$, (5.17) follows from (5.15) and the closedness of the gradient on $L^2(0, T; L^2)$.

Now, taking into account that by (5.17), with $q = 2p - 1 - \alpha$, we get

$$\int_\delta^t |\nabla(|u_\varepsilon|^{p-1})|_2^2 ds \le \|\mu\|(\nu(\delta,\mu))^{2(p-1)-\alpha}),\quad \forall t \ge \delta, \forall p \ge \frac{\alpha+3}{2}. \tag{5.18}$$

Moreover, by (5.14) we have $\{\nabla\beta(u_\varepsilon)\}_{\varepsilon>0}$ is bounded in $L^2(\delta, T, L^2)$ and this yields

$$\|\Delta\beta(u_\varepsilon) - \mathrm{div}(Db(u_\varepsilon)u_\varepsilon)\|_{L^2(\delta,T;H^{-1})} \le C,\ \forall\varepsilon > 0. \tag{5.19}$$

Note also that, by (5.18), it follows that

$$\int_\delta^t |\nabla(|u_\varepsilon|^{p-2}u_\varepsilon)|_2^2 ds \le C,\ \forall\varepsilon > 0. \tag{5.20}$$

Hence, $\{|u_\varepsilon|^{p-1}\}_{\varepsilon>0}$ is bounded in $L^2(\delta, T; H^1)$ and so, by (5.19), we infer that

$$\|\,|u_\varepsilon|^{p-1}(\Delta\beta(u_\varepsilon) - \mathrm{div}(Db(u_\varepsilon)u_\varepsilon))\|_{L^2(\delta,T;H^{-1})} \le C.$$

This implies that the set

$$\left\{\frac{\partial}{\partial t}(|u_\varepsilon|^{p-1}u_\varepsilon)\right\}_{\varepsilon>0} = \left\{p|u_\varepsilon|^{p-1}(\Delta\beta(u_\varepsilon) - \mathrm{div}(\nabla u_\varepsilon))\right\}_{\varepsilon>0}$$

is bounded in $L^2(\delta, T; H^{-1})$. Note that by (5.18) applied to $p + 1$ replacing p, we have that also $\{|u_\varepsilon|^{p-1}u_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^2(\delta, T; H^1)$.

Then, by the Aubin-Lions-Simon compactness theorem (see [21]), the set $\{|u_\varepsilon|^{p-1}u_\varepsilon\}_{\varepsilon>0}$ is relatively compact in $L^2(\delta, T; L^2_{\mathrm{loc}})$ for all $0 < \delta < T < \infty$. Hence, along a subsequence, we have for $\gamma(r) := |r|^{p-1}r$, $r \in \mathbb{R}$,

$$\gamma(u_\varepsilon) \to v \text{ a.e. on } (0,\infty) \times \mathbb{R}^d. \tag{5.21}$$

Then, since γ has a continuous inverse and since β is continuous, we have

$$u_\varepsilon \to u = \gamma^{-1}(v) \text{ and } \beta(u_\varepsilon) \to \beta(u),\ \text{a.e. on } (0,\infty) \times \mathbb{R}^d. \tag{5.22}$$

We also note that, by (5.12), (5.13), we have, for all $p \geq 1$, the estimate

$$\begin{aligned} \|u_\varepsilon(t)\|_{L^p(\mathbb{R}^d)} &\leq \|u_\varepsilon(t)\|_{L^1(\mathbb{R}^d)}^{\frac{1}{p}} \|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^d)}^{\frac{p-1}{p}} \\ &\leq C\|\mu\|^{\frac{2(p-1)}{(2+\alpha(d-1))p^2}}\, t^{-\frac{d(p-1)}{(2+(\alpha-1)d)p}},\ \forall t > 0,\ \varepsilon > 0. \end{aligned} \tag{5.23}$$

Since $\alpha > \frac{d-2}{d}$, we have $\alpha + \frac{2}{d} > 1$ and, for every $p \in \left[1, \alpha + \frac{2}{d}\right)$,we have

$$\frac{d(p-1)}{2 + (\alpha - 1)d} < 1. \tag{5.24}$$

Then, for such p, (5.23) implies that, for every $T > 0$,

$$\int_0^T \|u_\varepsilon(t)\|_{L^p(\mathbb{R}^d)}^p dt \leq C, \tag{5.25}$$

and, therefore, if in addition $p > 1$, along a subsequence $\varepsilon \to 0$,

$$u_\varepsilon \to u \text{ weakly in } L^p((0,T) \times \mathbb{R}^d). \tag{5.26}$$

Moreover, by (5.8), (5.22) and (5.25), it follows that $\{\beta(u_\varepsilon)\}$ is bounded in $L^q((0,T) \times \mathbb{R}^d)$ for all $q \in \left(1, 1 + \frac{2}{\alpha d}\right)$, and so (along a subsequence)

$$\beta(u_\varepsilon) \to \beta(u) \text{ weakly in } L^q((0,T) \times \mathbb{R}^d). \tag{5.27}$$

Since, by (2.9), we have

$$\int_0^\infty \int_{\mathbb{R}^d} (u_\varepsilon(\varphi_t + D \cdot \nabla\varphi) + \beta(u_\varepsilon)\Delta\varphi) dt\, dx + \int_{\mathbb{R}^d} \mu_\varepsilon(x)\varphi(0,x)dx = 0$$

for any $\varphi \in C_0^\infty([0,\infty) \times \mathbb{R}^d)$, letting $\varepsilon \to 0$, we see by (5.26) and (5.27) that u satisfies (5.2). As regards (5.4), (5.5), (5.6) and (5.7), these by (5.22) immediately follow from the corresponding properties of u_ε and (5.12). Furthermore, (5.8) follows from (5.25) and Fatou's Lemma.

Taking $p = \alpha$ in (5.8), (5.9) follows.

By (5.5) and (5.9), we may apply Lemma 8.1.2 in [1], to conclude that $t \mapsto u(t,x)dx \in \mathcal{M}_b$ has a $\sigma(\mathcal{M}_b, C_b)$-continuous version on $(0,\infty)$, denoted by μ_t, $t > 0$. To show (5.10), we apply (5.2) with $\varphi(t,x) = \psi(t)\zeta(x)$, $\psi \in C_0^\infty([0,\infty))$ and $\zeta \in C_0^\infty(\mathbb{R}^d)$. Then, for

$$L\zeta(t,x) = \beta(u(t,x))\Delta\zeta(x) + D(x) \cdot \nabla\zeta(x),$$

we have from (5.2)

$$\int_0^\infty \psi(t)\int_{\mathbb{R}^d} L\zeta\, d\mu_t\, dt + \psi(0)\int_{\mathbb{R}^d}\zeta\, d\mu = -\int_0^\infty \frac{d}{dt}\psi(t)\int_{\mathbb{R}^d}\zeta\, d\mu_t\, dt, \quad (5.28)$$

hence, choosing $\psi \in C_0^\infty((0,\infty))$, we obtain for dt-a.e. $t \in (0,\infty)$,

$$\int_{\mathbb{R}^d}\zeta\, d\mu_t = C + \int_0^t\int_{\mathbb{R}^d} L\zeta\, d\mu_s ds. \quad (5.29)$$

By (5.9), the right hand side is continuous in $t \in [0,\infty)$ and equal to C at $t = 0$, while, as seen above, also the left hand side is continuous in $t \in (0,\infty)$. Hence, we obtain that (5.29) holds for all $t \in (0,\infty)$ and

$$\lim_{t\to 0}\int_{\mathbb{R}^d}\zeta\, d\mu_t = C.$$

Plugging (5.29) into the right hand side of (5.28), with $\psi \in C_0^\infty([0,\infty))$ such that $\psi(0) = 1$ and integrating by parts, we find

$$\int_0^\infty \psi\int_{\mathbb{R}^d} L\zeta\, d\mu_t\, dt + \int_{\mathbb{R}^d}\zeta\, d\mu = C + \int_0^\infty \psi\int_{\mathbb{R}^d} L\zeta\, d\mu_t\, dt$$

and (5.10) follows, because (5.5) holds for all $t > 0$, as we shall see below.

It is obvious that, for the $\sigma(\mathcal{M}_b, C_b)$-continuous version $t \mapsto \mu_t$ of $t \mapsto u(t,x)dx$ on $(0,\infty)$, properties (5.4), (5.5) and (5.7) hold for all $t > 0$. For this version, it is also easily seen that

$$t \mapsto |u(t)|_\infty$$

is lower semicontinuous, hence also (5.6) follows for all $t > 0$.

It remains to prove the last assertion in Theorem 5.2. To express the dependence of our $\sigma(\mathcal{M}_b, C_b)$-continuous version $[0,\infty) \ni t \mapsto \mu_t \in \mathcal{M}_b$ with $\mu_0 = \mu$ of our solution to (5.2), we set, for $\mu \in \mathcal{M}_b$,

$$P(t)\mu = \mu_t, \quad t \geq 0,$$

and recall that μ_t has a density in L^1 for $t > 0$, which we identify with μ_t, i.e., $\mu_t \in L^1$, $\forall t > 0$. Let $T > 0$. By construction, we know that (along a subsequence depending on μ) $\varepsilon \to 0$

$$S(\cdot)(\mu*\rho_\varepsilon) \to P(\cdot)\mu, \text{ a.e. on } (0,T)\times\mathbb{R}^d \text{ and weakly in } L^p((0,T)\times\mathbb{R}^d) \quad (5.30)$$

as functions of (t,x) for $p \in \left(1, \alpha + \frac{2}{d}\right)$ (see (5.22), (5.26), respectively). Here $S(t)$, $t \geq 0$, is the semigroup from Theorem 5.2.

Claim. *If $\mu \in L^1$, then*

$$S(t)\mu = P(t)\mu \ \text{ for all } t \geq 0.$$

To prove the claim we recall that, since $\mu \in L^1$, we have $\mu * \rho_\varepsilon \to \mu$ in L^1. Hence, by (2.8) and (5.30),

$$S(t)\mu = P(t)\mu \text{ in } \mathcal{M}_b \text{ for } dt - \text{a.e. } t \in [0,T].$$

Since both sides are $\sigma(L^1, C_b)$-continuous in $t \in [0,T]$, this equality holds $\forall\, t \in [0,T]$, $T > 0$, and the Claim is proved.

Therefore, we may rename $P(t) : \mathcal{M}_b \to \mathcal{M}_b$, $t \geq 0$, and set $S(t) = P(t)$, $t \geq 0$, since it is an extension of $S(t) : L^1 \to L^1$ for every $t \geq 0$.

Finally, for $\mu, \widetilde{\mu} \in \mathcal{M}_b$ with corresponding solutions $u_\varepsilon, \widetilde{u}_\varepsilon$ to (5.11), we have by (2.8), for all $t \geq 0$,

$$|u_\varepsilon(t) - \widetilde{u}_\varepsilon(t)|_{L^1} \leq |(\mu - \widetilde{\mu}) * \rho_\varepsilon|_{L^1} \leq \|\mu - \widetilde{\mu}|_{\mathcal{M}_b}.$$

Hence, for all $\varphi \in C_b([0,\infty))$, $\varphi \geq 0$, by (5.22) and Fatous's lemma, letting $\varepsilon \to 0$ we get

$$\int_0^\infty \varphi(t)|S(t)\mu - S(t)\widetilde{\mu}|_{L^1} dt \leq \int_0^\infty \varphi(t)\|\mu - \widetilde{\mu}\|_{\mathcal{M}_b} dt,$$

so,

$$|S(t)\mu - S(t)\widetilde{\mu}|_{L^1} \leq \|\mu - \widetilde{\mu}\|_{\mathcal{M}_b} \text{ for } dt - \text{a.e. } t \in (0,\infty).$$

But the left hand side is lower semicontinuous in $t \in [0,\infty)$, since $t \mapsto S(t)\mu$ is $\sigma(L^1, C_b)$-continuous, hence

$$\|S(t)\mu - S(t)\widetilde{\mu}\|_{\mathcal{M}_b} \leq \|\mu - \widetilde{\mu}\|_{\mathcal{M}_b},\ \forall\, t \in [0,\infty).$$

Remark 5.3. One might suspect that, if $\mu \geq 0$, then under the hypotheses of Theorem 5.2 the solution u is the unique nonnegative solution to (5.2). This is true for the porous media equation ([20]). If the uniqueness is true for all $\mu \in L^1$, it follows by Theorem 2.2 that $u(t) \in C([\delta,\infty); L^1)$, for each $\delta > 0$. (In fact, $u(t)$ starting from δ would be the same on $[\delta,\infty)$ as the generalized solution to (1.1), which starts from δ with initial data $u(\delta) \in L^1 \cap L^\infty$.) In this case, our solutions in Theorem 5.2 starting from any $\mu \in \mathcal{M}_b$, would also have the flow property.

Remark 5.4. As shown in [14] for the special case $\beta(r) = |r|^{\alpha-1}$ and $D = 0$, no nonnegative distributional solution to (1.1) exists if $0 < \alpha < \frac{d-2}{d}$, $d \geq 3$, if μ is the Dirac measure, whereas for every $\mu \in \mathcal{M}_b$ a solution exists if $\alpha > \frac{d-2}{d}$, $d \geq 3$.

Remark 5.5. It should be noted that, as follows from the proof of Theorem 5.2, we have a further regularity property of $S(t)\mu$, $t \geq 0$, namely that, for every $\mu \in \mathcal{M}_b$, $p \geq \frac{\alpha+3}{2}$,

$$|S(\cdot)\mu|^{p-2}S(\cdot)\mu \in L^2(\delta, T; H^1),\ 0 < \delta < T < \infty.$$

Indeed, this follows from (5.20) and (5.22) by lower semicontinuity from the closedness of the gradient on $L^2(\delta, T; L^2)$.

Remark 5.6. If $D \equiv 0$, then, by the uniqueness result in [20], one can easily show that $S(t)$, $t \geq 0$, in Theorem 5.2 is, in fact, a semigroup on $\mathcal{M}_b$. We expect that this is also true for $D \not\equiv 0$ under the conditions of Theorem 5.2. This is a subject of our future study.

6 The McKean-Vlasov equation

As a direct consequence of Theorems 2.2 and 5.2, we obtain (probabilistically) weak solutions to the McKean-Vlasov SDE (1.2). More precisely, we have

Theorem 6.1. *Assume that one of the following holds:*

(a) *Hypotheses* (i), (ii), (iii) *from Section 1 and* (2.5) *hold. Furthermore, for some $m \geq 1$, we have $|\beta(r)| \leq c|r|^m$, $r \in \mathbb{R}$, and let u be the solution of* (2.9) *from Theorem* 2.2 *with the initial condition $\mu = u_0 dx$, $u_0 \in \mathcal{P}_0(\mathbb{R}^d) \cap L^\infty$.*

(b) *Hypotheses* (k), (kk), (kkk) *from Section* 4 *and* (5.3) *hold. Let u be the solution of* (5.2) *from Theorem* 5.2 *with the initial condition $\mu \in \mathcal{P}(\mathbb{R}^d)$.*

Then, there exists a (probabilistically) weak solution X to (1.2) *on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ with an $\mathbb{R}^d$-valued $(\mathcal{F}_t)$-Brownian motion $W(t)$, $t \geq 0$, such that*

$$\mu = \mathbb{P} \circ (X(0))^{-1} \quad \textit{and} \quad u(t,x)dx = \mathbb{P} \circ (X(t))^{-1}(dx),\ \forall t > 0. \tag{6.1}$$

Proof. By our assumptions, we have, for every $T > 0$,

$$\int_0^T \int_{\mathbb{R}^d} (|\beta(u(t,x))| + |b(u(t,\cdot)| \, |D(x)| u(t,x)) dx \, dt < \infty.$$

Hence, the assertion follows immediately by Section 2 in [6]. □

Appendix

Lemma A.1. *Let* $\alpha \in \left(\frac{d-2}{2}, \infty\right)$, $p_0 \in (1, \infty)$. *Let* $C_{\alpha,d}$ *and* γ *be as defined in* (4.15), (4.17), *respectively. Then*

(i) $\gamma = 1 - \dfrac{(p_0 - 1)(d-2)}{(p_0 + \alpha - 2)d + 2}$. *Hence* $\gamma = (0, 1)$.

(ii) $\gamma + \alpha - 1 > 0$.

(iii) *If* $p_0 < C_{\alpha,d}$, *Then* $\dfrac{\gamma - p_0}{\gamma + \alpha - 1} > -1$.

Proof. (i) We have

$$(p_0 + \alpha - 2)d + 2 = (p_0 - 1)(d-2) + d(\alpha - 1) + 2p_0, \tag{A.1}$$

which implies the formula for γ. Noting that, by assumption on α, we have

$$d(p_0 + \alpha - 2) + 2 > (p_0 - 1)d > 0,$$

we conclude that $\gamma < 1$.

Furthermore, by assumption on α and since $p_0 > 1$,

$$d(\alpha - 1) + 2p_0 > -2 + 2p_0 > 0,$$

hence $\gamma > 0$ due to (A.1).

(ii) By (i) we have to show that

$$\alpha > \frac{(p_0 - 1)(d-2)}{d(p_0 + \alpha - 2) + 2}.$$

But this is equivalent to

$$\alpha \left(\frac{2}{d} + p_0 - 2 \right) + \alpha^2 > (p_0 - 1) \left(1 - \frac{2}{d} \right),$$

which in turn is obviously true, since by assumption on α,

$$(p_0-1)\left(1-\frac{2}{d}\right) < (p_0-1)\alpha + \alpha\left(\alpha - \left(1-\frac{2}{d}\right)\right).$$

(iii) By (4.17) and (ii), we have to show

$$(\alpha-1-p_0)(d(\alpha-1+p_0)+2-d)+2(2p_0+(\alpha-1)d) > 0.$$

But this is equivalent to

$$d(\alpha-1)^2 - dp_0^2 + (\alpha-1)(2-d) - p_0(2-d) + 4p_0 + 2d(\alpha-1) > 0,$$

which means

$$-p_0^2 + \frac{(2+d)}{d}\,p_0 > -\frac{\alpha-1}{d}\,(d(\alpha-1)+d+2),$$

i.e.,

$$\left(p_0 - \frac{d+2}{2d}\right)^2 < (\alpha-1)\left(\alpha+\frac{2}{d}\right) + \left(\frac{d+2}{2d}\right)^2.$$

But the latter holds, if $p_0 < C_{\alpha,d}$.

Lemma A.2. *Consider the situation of Lemma* A.1. *Then*

(i) $p_0 - \gamma = \frac{(p_0-1)(p_0+\alpha-1)d}{(p_0+\alpha-2)d+2}$.

(ii) $\gamma + \alpha - 1 = \frac{(2+(\alpha-1)d)(p_0+\alpha-1)}{(p_0+\alpha-2)d+2}$.

(iii) $\dfrac{2\gamma(p_0+\alpha-1)}{(\gamma+\alpha-1)(2p_0+(\alpha-1)d)} = \dfrac{2}{2+(\alpha-1)d}$.

Proof. (i) By Lemma A.1 (i), we have

$$p_0 - \gamma = (p_0-1)\left(1+\frac{d-2}{(p+\alpha-2)d+2}\right),$$

from which (i) follows. (ii) By (i), we have

$$\begin{aligned}\gamma+\alpha-1 &= p_0+\alpha-1-\frac{(p_0-1)(p_0+\alpha-1)d}{(p_0+\alpha-2)d+2}\\ &= (p_0+\alpha-1)\left(1-\frac{(p_0-1)d}{(p_0+\alpha-2)d+2}\right),\end{aligned}$$

which implies (i).

(iii) follows from (4.17) and (ii).

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