OCTAV MAYER INSTITUTE OF MATHEMATICS
ROMANIAN ACADEMY, IAŞI BRANCH
Bd. Carol I nr. 10, 700506, Iaşi, Romania

PREPRINT SERIES OF THE
OCTAV MAYER INSTITUTE OF MATHEMATICS

Uniqueness for nonlinear Fokker-Planck equations and weak uniqueness for McKean-Vlasov SDEs

Viorel Barbu
Octav Mayer Institute of Mathematics of the Romanian Academy, Iaşi, Romania

Michael Röckner
Fakultät für Mathematik, Universität Bielefeld, D33501, Germany
Academy of Mathematics and System Sciences, CAS, Beijing

Nr. 04-2020

Uniqueness for nonlinear Fokker–Planck equations and weak uniqueness for McKean-Vlasov SDEs

Viorel Barbu* Michael Röckner†‡

Abstract

One proves the uniqueness of distributional solutions to nonlinear Fokker–Planck equations with monotone diffusion term and derive as a consequence (restricted) uniqueness in law for the corresponding McKean–Vlasov stochastic differential equation (SDE).

Mathematics Subject Classification (2000): 60H30, 60H10, 60G46, 35C99.

Keywords: Fokker–Planck equation, mild solution, distributional solution.

1 Introduction

Consider the nonlinear Fokker–Planck equation

$$\begin{aligned} &u_t - \Delta\beta(u) + \mathrm{div}(b(x,u)u) = 0 \text{ in } \mathcal{D}'((0,\infty)\times\mathbb{R}^d),\\ &u(0,x) = u_0(x), \end{aligned} \tag{1.1}$$

where $\beta : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ satisfy the following assumptions

*Octav Mayer Institute of Mathematics of the Romanian Academy, Iaşi, Romania. Email: vbarbu41@gmail.com

†Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany. Email: roeckner@math.uni-bielefeld.de

‡Academy of Mathematic and System Sciences, CAS, Beijing, China

(i) $\beta(0)=0$, $\beta\in C^1(\mathbb{R})$, *and*

$$\gamma_0|r_1-r_2|^2\le(\beta(r_1)-\beta(r_2))(r_1-r_2),\ \ r_1,r_2\in\mathbb{R}, \tag{1.2}$$

where $0<\gamma_0<\infty$.

(ii) $b^i\in C_b(\mathbb{R}^d)$, $b^i(x,0)\equiv 0$, $x\in\mathbb{R}^d$, $i=1,2,...,d$,

$$\sup\{|b^i_r(x,r)|; x\in\mathbb{R}^d,\ i=1,2,...,d,\ |r|\le M\}\le C_M,\ \forall M>0,$$

and, for

$$\delta(r):=\sup\{|b_x(x,r)|;\ x\in\mathbb{R}^d\},$$

we have $\delta\in C_b(\mathbb{R})$.

Here

$$b(x,u)=\{b^i(x,u)\}_{i=1}^d \text{ and } b^i_r=\frac{\partial b^i}{\partial r},\ b_x=\left\{\nabla_x b^i(x,\cdot)\right\}_{i=1}^d.$$

Equation (1.1) is relevant in statistical mechanics, kinetic theory as well in theory of stochastic differential equations.

By a distributional solution (in the sense of Schwartz) with initial condition $u_0\in L^1$ we mean a function $u:[0,\infty)\to L^1(\mathbb{R}^d)$ such that $(u(t,\cdot)dx)_{t\in[0,T]}$ is narrowly continuous, that is,

$$\lim_{t\to s}\int_{\mathbb{R}^d}u(t,x)\psi(x)dx=\int_{\mathbb{R}^d}u(s,x)\psi(x)dx,\ \forall\psi\in C_b(\mathbb{R}^d),\ s\ge 0, \tag{1.3}$$

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^d}(u(t,x)\varphi_t(t,x)+\beta(u(t,x))\Delta\varphi(t,x) \qquad (1.4)\\ &\quad +b(x,u(t,x))u(t,x)\cdot\nabla_x\varphi(t,x))dt\,dx=0,\\ &\qquad\qquad\qquad \forall\varphi\in C_0^\infty((0,\infty)\times\mathbb{R}^d)\end{aligned}$$

$$u(0,x)=u_0(x),\ \text{ a.e. } x\in\mathbb{R}^d. \tag{1.5}$$

(In the following, we shall use the notation $b^*(x,u)=b(x,u)u$.)

The existence of a weak (generalized) solution to the Fokker–Planck equation (1.1) which under the above assumptions is also a distributional solution, was studied under different sets of hypotheses on β and b in the authors works [2]–[4]. For instance, in [2] it was proved, in particular, that, if (i)–(ii) hold and, in addition, for $\Phi(u)\equiv\frac{\beta(u)}{u}$, $u\in\mathbb{R}$, we have $\Phi\in C^2(\mathbb{R})$, then there is a mild solution $u \in C([0,\infty); L^1(\mathbb{R}^d))$ for each $u_0 \in L^1(\mathbb{R}^d)$. The mild solution u is defined as

$$u(t) = \lim_{h\to 0} u_h(t) \text{ in } L^1(\mathbb{R}^d),\ \forall t \geq 0,$$

where u_h is defined by the finite difference scheme

$$\begin{aligned}
&u_h(t) = u_h^i \text{ for } t \in [ih, (i+h)h],\ i = 0, 1, ..., Nh = T, \\
&u_h^{i+1} - h\Delta\beta(u_h^{i+1}) + h\,\mathrm{div}(b(x, u_h^{i+1})u_h^{i+1}) = u_h^i \text{ in } \mathcal{D}'(\mathbb{R}^d), \\
&\hspace{18em} i = 0, 1, ..., \\
&u_h^0 = u_0.
\end{aligned}\tag{1.6}$$

Moreover, $S(t)u_0 = u(t),\ t \geq 0$, is a strongly continuous semigroup of nonexpansive mappings in $L^1(\mathbb{R}^d)$ which leaves invariant the set $\mathcal{P}$ of all probability densitis functions, that is,

$$\mathcal{P} = \left\{\rho \in L^1(\mathbb{R}^d);\ \rho \geq 0, \text{ a.e. on } \mathbb{R}^d, \int_{\mathbb{R}^d} \rho\, dx = 1\right\}.$$

The idea of the proof is to represent equation (1.1) as the Cauchy problem in the space $L^1(\mathbb{R}^d)$

$$\begin{aligned}
&\frac{du}{dt} + Au = 0,\ t \geq 0, \\
&u(0) = u_0,
\end{aligned}\tag{1.7}$$

where A is an m-accretive realization of the operator $u \to -\Delta\beta(u) + \mathrm{div}(b(x,u)u)$ in the space $L^1(\mathbb{R}^d)$. Then, by the Crandall & Liggett existence theorem (see [1], p. 131), it follows the existence of a unique mild solution u which is defined by (1.5)-(1.6).

In [5], it is proved the existence of a generalized solution in sense of (1.6) in the special case where $0 < \gamma \leq \beta'(r) \leq \gamma_1$, $\forall r \in \mathbb{R}$, and $b(x,u) \equiv D(x)b(u)$, $D = -\nabla\Phi$, $\Phi \in C^1(\mathbb{R}^d)$, $\lim\limits_{|x|\to\infty} \Phi(x) = +\infty$. (This is the nondegenerate-conservative case.) One proves, in addition, that if $b(r) \geq b_0 > 0$, $\forall r \geq 0$, and

$$\gamma_1\Delta\Phi - b_0|\nabla\Phi|^2 \leq 0 \ \text{ in } \mathbb{R}^d,$$

then is true for equation (1.1) the so called H-theorem, that is,

$$\lim_{t\to\infty} u(t) = u_\infty \ \text{ in } L^1(\mathbb{R}^d),$$

for $u_0 \in \mathcal{P} \cap \mathcal{M}$, where u_∞ is the unique solution in $\mathcal{M} \cap \mathcal{P}$ to the corresponding steady state equation. Here, $\mathcal{M} = \left\{ u \in L^1(\mathbb{R}^d); \int_{\mathbb{R}^d} \Phi(x) u(x) dx < \infty \right\}$. In [4], it is proved for $u_0 \in L^1(\mathbb{R}^d)$ the existence of a generalized (mild) solution $u \in C([0,\infty); L^1(\mathbb{R}^d)$ in the (eventually) degenerate case, where

$$\beta' \geq 0,\ D \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d),\ \mathrm{div}(D) \in L^\infty(\mathbb{R}^d),$$

and $b \in C^1(\mathbb{R})$ is bounded, nonnegative and $b \equiv$ const. if $(\mathrm{div}\ D)^- \not\equiv 0$ or β is not strictly increasing.

Moreover, if $\beta \in C^2(\mathbb{R})$, $\beta'(r) \geq a|r|^{\alpha-1}$, $\beta(0) = 0$, where $\alpha \geq 1$, $d \geq 3$, $a > 0$, and

$$D \in (L^2 \cap L^\infty)(\mathbb{R}^d; \mathbb{R}^d), \mathrm{div}\, D \in L^\infty(\mathbb{R}^d), \mathrm{div}\, D \geq 0,\ b \in C_b(\mathbb{R}) \cap C^1(\mathbb{R}),\ b \geq 0,$$

then the existence results extends to all measure initial data $u_0 \geq 0$. More precisely, there is a distributional solution $u = u(t,x)$ to (1.1) which has a $\sigma(\mathcal{M}_b, C_b)$-continuous version on $(0,\infty)$. (Here $\mathcal{M}_b$ is the space of bounded Radon measures on $\mathbb{R}^d$.)

It should be emphasized that in all the situations presented above the mild (generalized) solution u to (1.1) is shown to be unique in the class of mild solutions defined by (1.6), but not in the class of distributional solutions defined by (1.3)-(1.5). The uniqueness of distributional solutions u in (1.1) remains an important objective for its implications, which so far was fulfilled for some special classes of porous media equations only ([4], [6]) and in [7] for a special class of Fokker–Planck equations.

Here, we shall prove under hypotheses (i), (ii) the uniqueness for (1.1) in the class of distributional solutions and derive from this result the uniqueness in law of solutions to McKean–Vlasov SDE

$$dX(t) = b(X(t), u(t, X(t)))dt + \frac{1}{\sqrt{2}} \left(\frac{\beta(u(t, X(t))}{u(t, X(t))} \right)^{\frac{1}{2}} dW(t). \tag{1.8}$$

Notation. Denote by $L^p(\mathbb{R}^d) = L^p$ the space of p-summable functions on L^p, with the norm denoted $|\cdot|_p$. By $H^k(\mathbb{R}^d) = H^k$, $k = 1, 2$, and $H^{-k}(\mathbb{R}^d) = H^{-k}$, we denote the standard Sobolev spaces on $\mathbb{R}^d$ and by $C_b(\mathbb{R}^d)$ the space of continuous and bounded functions on $\mathbb{R}^d$. By $C^k(\mathbb{R}^d)$ we denote the space of continuously differentiable functions on $\mathbb{R}^d$ of order k, by $C_b^1(\mathbb{R}^d)$ the space $\left\{ u \in C^1(\mathbb{R}^d); \nabla_x \in C_b(\mathbb{R}^d),\ j = 1, ..., d \right\}$. We also set

$$\nabla_x u = \left\{ \frac{\partial u}{\partial x_i} \right\}_{i=1}^d, \quad \Delta u = \sum_{i=1}^d \frac{\partial^2 u}{\partial x_i^2}.$$

The spaces of continuous and differentiable functions on $(0,T)\times\mathbb{R}^d$ are denoted in a similar way and we shall simply write

$$C_b^1(\mathbb{R}^d)=C_b^1,\ C^k(\mathbb{R}^d)=C^k,\ k=1,2.$$

The scalar product in L^2 is denoted $\langle\cdot,\cdot\rangle_2$ and by ${}_{H^{-1}}\langle\cdot,\cdot\rangle_{H^1}$ the pairing between H^1 and H^{-1}. Of course, on $L^2\times L^2$ this coincides with $\langle\cdot,\cdot\rangle_2$. The scalar product $\langle\cdot,\cdot\rangle_{-1}$ on H^{-1} is taken as

$$\langle u,v\rangle_{-1}=((I-\Delta)^{-1}u,v)_2,\ \forall u,v\in H^{-1} \tag{1.9}$$

with the corresponding norm

$$|u|_{-1}=(\langle u,u\rangle_{-1})^{\frac{1}{2}},\ u\in H^{-1}. \tag{1.10}$$

By $\mathcal{D}'((0,\infty)\times\mathbb{R}^d)$ and $\mathcal{D}'(\mathbb{R}^d)$ we denote the space of Schwartz distributions on $(0,\infty)\times\mathbb{R}^d$ and $\mathbb{R}^d$, respectively. If $\mathcal{X}$ is a Banach space, we denote by $W^{1,2}([0,T];\mathcal{X})$ the infinite dimensional Sobolev space $\{y\in L^2(0,T;\mathcal{X});\ \frac{dy}{dt}\in L^2(0,T;\mathcal{X})\}$, where $\frac{d}{dt}$ is taken in the sense of vectorial distributions. We also set, for each $z\in C^1(\mathbb{R}^d\times\mathbb{R})$,

$$z_r(x,r)=\frac{\partial}{\partial r}\,z(x,r),\quad z_x=\nabla_x z(x,r).$$

We shall denote the norms on $\mathbb{R}^d$ and $\mathbb{R}$ by the same symbol $|\cdot|$.

2 The main result

The next result is a uniqueness theorem for distributional solutions u to (1.1). In the special case $b\equiv 0$, such a uniqueness result for(1.1) was established earlier in [4] for continuous and monotonically nondecreasing functions β. (See, also, [9].)

Theorem 2.1. *Let $T>0$ and let conditions* (i)–(ii) *on β and b hold. Then, for each $u_0\in L^\infty\cap L^1$, the Fokker–Planck equation* (1.1) *has at most one distributional solution $u\in L^\infty((0,T);L^1)\cap L^\infty((0,T)\times\mathbb{R}^d)$.*

Proof. Let $u_1,u_2\in L^\infty(0,T;L^1)\cap L^\infty((0,T)\times\mathbb{R}^d)$ be two distributional solutions to (1.1) and let $u=u_1-u_2$. We have

$$\begin{aligned}&u_t-\Delta(\beta(u_1)-\beta(u_2))+\operatorname{div}(b^*(x,u_1)-b^*(x,u_2))=0\\&\qquad\qquad\qquad\qquad\qquad\qquad\text{in }\mathcal{D}'((0,\infty)\times\mathbb{R}^d)\\&u(0,x)=0.\end{aligned} \tag{2.1}$$

(Here, $b^*(x,r) = b(x,r)r,\ \forall x \in \mathbb{R}^d,\ r \in \mathbb{R}$.)

It should be mentioned that, by (1.1), it follows that $u_i, \beta(u_i) \in L^2((0,T);L^2)$, $i = 1,2$, and, therefore, $u \in W^{1,2}([0,T];H^{-2})$.

Consider the operator $\Gamma : H^{-1} \to H^1$ defined by

$$\Gamma u = (1-\Delta)^{-1}u,\ u \in H^{-1}(\mathbb{R}^d)$$

and note that Γ is an isomorphism of H^{-1} onto H^1, and also that $\Gamma \in L(H^{-2}, L^2)$. Since $u_i \in L^2(0,T;L^2)$, $i = 1,2$, it follows that $y = \Gamma u \in L^2(0,T;H^2) \cap W^{1,2}([0,T];L^2)$ and so, by (2.1), we have

$$\begin{aligned} &\frac{dy}{dt} - \Gamma\Delta(\beta(u_1) - \beta(u_2)) + \Gamma \operatorname{div}(b^*(x,u_1) - b^*(x,u_2)) = 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } t \in (0,T), \\ &y(0) = 0, \end{aligned} \tag{2.2}$$

where $\frac{dy}{dt} \in L^2(0,T;L^2)$. (We note that here $\frac{dy}{dt}$ is taken in the sense of L^2-valued vectorial distributions on $(0,T)$ and so $y : [0,T] \to L^2$ is absolutely continuous.) Hence, $u : [0,T] \to H^{-2}$ is absolutely continuous.

Now, we take the scalar product in L^2 of (2.2) with $u = u_1 - u_2$. Taking into account that

$$\left\langle \frac{dy}{dt}(t), y(t) \right\rangle_2 = \frac{1}{2}\,\frac{d}{dt}\,|y(t)|_2^2,\ \text{a.e. } t \in (0,T),$$

we get, by (1.9)–(2.2) that

$$\begin{aligned} \frac{1}{2}\,\frac{d}{dt}\,|u(t)|_{-2}^2 + \langle \beta(u_1) - \beta(u_2), u_1 - u_2\rangle_2 &= \langle \Gamma(\beta(u_1) - \beta(u_2)), u_1 - u_2\rangle_2 \\ - \langle \Gamma \operatorname{div}((b^*(x,u_1) - b^*(x,u_2)), u_1 - u_2\rangle_2\,,&\ \text{a.e. } t \in (0,T), \end{aligned}$$

where $|\cdot|_{-2}$ is the norm of H^{-2}. By (1.2), this yields

$$\begin{aligned} \frac{1}{2}\,\frac{d}{dt}\,|u(t)|_{-2}^2 + \gamma_0 |u(t)|_2^2 &\le \langle \beta(u_1(t)) - \beta(u_2(t)), u_1(t) - u_2(t)\rangle_{-1} \\ &\quad - \langle \operatorname{div}((b^*(x,u_1(t)) - b^*(x,u_2(t))), u_1 - u_2\rangle_{-1}\,. \end{aligned} \tag{2.3}$$

We note that

$$|\Gamma f|_2 \le |f|_2,\ \forall f \in L^2,$$

and, therefore,

$$|f|_{-1} \le |f|_2,\ \forall f \in L^2. \tag{2.4}$$

We also have

$$|\mathrm{div}\, F|_{-1} \le 2|F|_2,\ \ \forall F \in (L^2)^d. \tag{2.5}$$

This yields

$$\begin{aligned} |\left\langle \beta(u_1) - \beta(u_2), u_1 - u_2 \right\rangle_{-1}| &\le |\beta(u_1) - \beta(u_2)|_2 |u|_{-1} \\ &\le \beta_M |u|_2 |u|_{-1} \le \beta_M |u|_2^{\frac{3}{2}}\ |u|_{-2}^{\frac{1}{2}} \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} &\left|\langle \mathrm{div}\ (b^*(x,u_1) - b^*(x,u_2)), u_1 - u_2 \rangle_{-1}\right| \\ &\quad \le 2|(b^*(x,u_1) - b^*(x,u_2))|_2 |u_1 - u_2|_{-1} \\ &\quad \le 2(|b|_\infty + b_M |u_1|_\infty)|u|_2 |u|_{-1} \\ &\quad \le 2(|b|_\infty + b_M |u_1|_\infty)|u|_2^{\frac{3}{2}} |u|_{-2}^{\frac{1}{2}}. \end{aligned} \tag{2.7}$$

where $M = \max\{|u_1|_\infty, |u_2|_\infty)$ and

$$\begin{aligned} \beta_M &= \sup\{\beta'(r);\ |r| \le M\}, \\ b_M &= \sup\left\{\frac{|b(x,u_1) - b(x,u_2)|}{|r_1 - r_2|};\ x \in \mathbb{R}^d,\ |r_1|, |r_2| < M\right\} \\ &\le \sup\{|b_r(x,r)|; |r| \le M,\ x \in \mathbb{R}^d\}. \end{aligned}$$

(Here, we have used the interpolation inequality $|u|_{-1} \le |u|_2^{\frac{1}{2}} |u|_{-2}^{\frac{1}{2}}$.)

By (2.3)–(2.7), we get

$$\frac{1}{2}\ \frac{d}{dt}\,|u(t)|_{-2}^2 + \gamma_0 |u(t)|_2^2 \le (\beta_M + 2(|b|_\infty + b_M |u_1|_\infty)|u(t)|_2^{\frac{3}{2}} |u(t)|_{-2}^{\frac{1}{2}}, \qquad \text{a.e. } t \in (0,T),$$

where $|b|_\infty = \sup\{|b(x,r)|; x \in \mathbb{R}^d, r \in \mathbb{R}\}$. This yields

$$\frac{d}{dt}\,|u(t)|_{-2}^2 \le C|u(t)|_{-2}^2,\ \text{ a.e. } t \in (0,T).$$

Since $u : [0,T] \to H^{-2}$ is absolutely continuous and narrowly continuous, we infer that $|u(t)|_{-2} = 0$, $\forall t \in [0,T]$, and so $u \equiv 0$, as claimed. $\square$

As mentioned earlier, under hypotheses (i)–(ii), if $\Phi \in C^2$, where $\Phi(u) \equiv \frac{\beta(u)}{u}$, $u \in \mathbb{R}$, then the Fokker–Planck equation (1.1), for each $u_0 \in L^1$, has a unique mild solution $u \in C([0,\infty); L^1)$. This mild solution is also easily checked to be a distributional solution to (1.1). As regards this solution, we also have

Proposition 2.2. *Assume that* (i)–(ii) *hold, and that, for* $\Phi(u) \equiv \frac{\beta(u)}{u}$,

(iii) $\Phi \in C^2(\mathbb{R}^d)$.

Then, for each $u_0 \in L^1 \cap L^\infty$*, the mild solution* u *to* (1.1) *satisfies also*

$$u \in L^\infty((0,T) \times \mathbb{R}^d),\ \forall T > 0. \tag{2.8}$$

Proof. We rewrite (1.1) as

$$\begin{aligned}
&(u - |u_0|_\infty - \alpha(t))_t - \Delta(\beta(u) - \beta(|u_0|_\infty + \alpha(t)))\\
&\qquad + \mathrm{div}(b^*(x,u) - b^*(x, |u_0|_\infty + \alpha(t)))\\
&\qquad = -\mathrm{div}(b^*(x, |u_0|_\infty + \alpha(t))) - \alpha'(t) \le 0 \text{ in } (0,\infty) \times \mathbb{R}^d,
\end{aligned} \tag{2.9}$$

where $\alpha \in C^1([0,\infty))$ is chosen in such a way that

$$\begin{aligned}
&\alpha'(t) + \sup\{|b_x(x, |u_0|_\infty + \alpha(t))|; x \in \mathbb{R}^d\}(|u_0|_\infty + \alpha(t)) = 0,\ t \in (0,T),\\
&\alpha(0) = 0.
\end{aligned} \tag{2.10}$$

We may find α of the form $\alpha = \eta - |u_0|_\infty$, where η is a solution to the equation

$$\begin{aligned}
&\eta' - \delta(\eta)\eta = 0,\ t \ge 0,\\
&\eta(0) = |u_0|_\infty,
\end{aligned} \tag{2.11}$$

$\delta(r) = \sup\{|b_x(x,r)|; x \in \mathbb{R}^d\}$, $r \in \mathbb{R}$. Clearly, (2.11) has such a solution $\eta \in C^1([0,\infty))$, $\eta \ge 0$, on $[0,\infty)$ because $\delta \in C_b(\mathbb{R})$.

Formally, if we multiply (2.9) by $\mathrm{sign}(u - |u_0|_\infty - \alpha)^+$, integrate over $\mathbb{R}^d$ and use the monotonicity of β, we get by (2.5) that

$$\frac{d}{dt}\,|(u(t) - |u_0|_\infty - \alpha(t))^+|_1 \le 0,\ \text{a.e. } t \in (0,T). \tag{2.12}$$

This yields $u(t) \le |u_0|_\infty + \alpha(t)$, $\forall t \ge 0$, and similarly it follows that $u(t) \ge -|u_0|_\infty - \alpha(t)$. Hence, $u \in L^\infty((0,T) \times \mathbb{R}^d)$, as claimed.

The above formal argument can be made rigorous if u is a strong solution to (1.1) (which is not the case here). Then (see the detailed argument in [2], [4], [5])

$$\begin{aligned}
&\lim_{\delta \to 0} \frac{1}{\delta} \int_{[0 < (\beta(u) - \beta(|u_0|_\infty + \alpha(t))^+) \le \delta]} |b^*(x,u) - b^*(x, |u_0|_\infty + \alpha(t))|\,|\nabla u| dx\\
&\quad = \lim_{\delta \to 0} \frac{1}{\delta} \int_{[0 < (\beta(u) - \beta(|u_0|_\infty + \alpha(t))^+) \le \delta]} (|b(x,u) - b(x, |u_0|_\infty + \alpha(t))|\,|u|\\
&\quad + |b(x, |u_0|_\infty + \alpha(t))|\,|u - |u_0|_\infty - \alpha(t)|)|\nabla u| dx = 0,\ \forall t \in (0,T),
\end{aligned} \tag{2.13}$$

which is true if $\nabla u \in L^2(0,T;L^2)$ and $b(x,\cdot) \in \mathrm{Lip}(\mathbb{R})$ uniformly in x (which is the case if $b_r \in C_b(\mathbb{R}^d \times \mathbb{R})$). In order to be in such a situation, we approximate (1.1) by

$$\begin{aligned} & u_t - \Delta(\beta(u) + \varepsilon\beta(u) + \operatorname{div}(b_\varepsilon(x,u)u)) = 0 \text{ in } (0,T)\times\mathbb{R}^d, \\ & u(0,x) = u_0(x), \end{aligned} \tag{2.14}$$

where $\varepsilon > 0$ and $b_\varepsilon \in C_b^1(\mathbb{R}^d \times \mathbb{R})$ is a smooth approximation of b. (For instance, $b_\varepsilon = b * \rho_\varepsilon$, where ρ_ε is a standard mollifier.) Then, as proved earlier in [2], [3], [5], equation (2.14) has a unique solution $u_\varepsilon \in L^2(0,T;H^1) \cap C([0,T];L^1) \cap W^{1,2}([0,T];H^{-1})$ and $u_\varepsilon \to u$ in $C([0,T];L^1)$ as $\varepsilon \to 0$. An easy way to prove this is to apply the Trotter–Kato theorem (see [1], p. 169) to the family of m-accretive operators in L^1

$$\begin{aligned} A_\varepsilon u &= -\Delta\beta(u) + \varepsilon\beta(u) + \operatorname{div}(b_\varepsilon(x,u)u), \\ D(A_\varepsilon) &= \{u \in L^1; -\Delta\beta(u) + \varepsilon\beta(u) + \operatorname{div}(b_\varepsilon(x,u)u) \in L^1\}. \end{aligned}$$

(See the argument in [5].) Then, we replace (2.9) by

$$\begin{aligned} & (u_\varepsilon - |u_0|_\infty - \alpha(t))_t - \Delta(\beta(u_\varepsilon) - \beta(|u_0|_\infty + \alpha(t))) \\ & + \varepsilon(\beta(u) - \beta(|u_0|_\infty + \alpha(t))) + \operatorname{div}(b_\varepsilon^*(x,u_\varepsilon) - b_\varepsilon^*(x,|u_0|_\infty + \alpha(t)) \\ & = -b_\varepsilon^*(x,|u_0|_\infty + \alpha(t)) - \alpha'(t) - \varepsilon\beta(|u_0|_\infty + \alpha(t)) \le 0, \\ & \hspace{6cm} \text{a.e. in } (0,T)\times\mathbb{R}^d, \end{aligned} \tag{2.15}$$

where $b_\varepsilon^*(u) = b_\varepsilon(u)u$.

Let $\mathcal{X}_\delta \in \mathrm{Lip}(\mathbb{R})$ be the following approximation of the signum function

$$\mathcal{X}_\delta(r) = \begin{cases} 1 & \text{for} \qquad r \ge \delta, \\ \dfrac{r}{\delta} & \text{for} \qquad |r| < \delta, \\ -1 & \text{for } r < -\delta, \end{cases}$$

where $\delta > 0$. If we multiply (2.15) by $\mathcal{X}_\delta((\beta(u_\varepsilon) - \beta(|u_0|_\infty + \alpha))^+)$ and integrate over $\mathbb{R}^d$, we get

$$\begin{aligned} & \int_{\mathbb{R}^d} (u_\varepsilon - |u_0|_\infty - \alpha)_t \mathcal{X}_\delta((\beta(u_\varepsilon) - \beta(|u_0|_\infty + \alpha))^+) dx \\ & \quad \le \frac{1}{\delta} \int_{[0<(\beta(u_\varepsilon)-\beta(|u_0|_\infty+\alpha))^+\le\delta]} (b^*(x,u_\varepsilon)u_\varepsilon - b_\varepsilon^*(x,|u_0|_\infty + \alpha)) \cdot \nabla u_\varepsilon \, dx, \\ & \hspace{10cm} \forall t \in (0,T), \end{aligned}$$

because β is monotonically increasing and

$$\nabla(\beta(u_\varepsilon) - \beta(|u_0|_\infty + \alpha) \cdot \nabla \mathcal{X}_\delta((\beta(u_\varepsilon) - \beta(|u_0|_\infty + \alpha))^+) \geq 0 \text{ in } (0,T) \times \mathbb{R}^d.$$

Then, by (2.13), we get, for $\delta \to 0$,

$$\int_{\mathbb{R}^d} (u_\varepsilon - |u_0|_\infty - \alpha(t))_t^+ \, dx \leq 0, \ \forall t \in (0,T),$$

and this yields

$$u_\varepsilon(t,x) - |u_0|_\infty - \alpha(t) \leq 0, \text{ a.e. on } (0,T) \times \mathbb{R}^d,$$

and so, $u_\varepsilon \leq |u_0|_\infty + \alpha$, a.e. on $(0,T) \times \mathbb{R}^d$. Then, we pass to the limit $\varepsilon \to 0$ to get the claimed inequality. $\square$

By Theorem 2.1 and Proposition 2.2, we therefore get the following existence and uniqueness result for (1.1).

Theorem 2.3. *Under hypotheses* (i)–(iii)*, for each $u_0 \in L^1 \cap L^\infty$, equation* (1.1) *has a unique distributional solution*

$$u \in L^\infty((0,T); L^1) \cap L^\infty((0,T) \times \mathbb{R}^d), \ \forall T > 0. \tag{2.16}$$

3 The uniqueness of the linearized equation

Consider a distributional solution of the linearized equation corresponding to (1.1), that is,

$$\begin{aligned} &v_t - \Delta(\Phi(u)v + \operatorname{div}(b(x,u)v)) = 0 \text{ in } \mathcal{D}'((0,\infty) \times \mathbb{R}^d), \\ &v(0,x) = v_0(x), \end{aligned} \tag{3.1}$$

where $u \in L^\infty((0,T) \times \mathbb{R}^d), \ \forall T > 0$. By (i)–(ii), we have

$$b(x,u), \Phi(u) = \frac{\beta(u)}{u} \in L^\infty((0,\infty) \times \mathbb{R}^d).$$

Moreover, we have

$$\Phi(u) \geq \gamma_0 > 0, \text{ a.e. in } (0,\infty) \times \mathbb{R}^d. \tag{3.2}$$

In the following, we denote $\Phi(u(t,x))$ by $\Psi(t,x)$, $(t,x) \in (0,\infty) \times \mathbb{R}^d$.

Theorem 3.1. (Linearized uniqueness) *Under hypotheses* (i)–(ii)*, for each $v_0 \in L^1 \cap L^\infty$ and $T > 0$, equation* (3.1) *has at most one distributional solution $v \in C([0,T];L^1) \cap L^\infty((0,T)\times\mathbb{R}^d)$.*

Proof. We shall proceed as in the proof of Theorem 2.1. Namely, we set $v_1 - v_2 = v$ for two solutions v_1, v_2 of (3.2) and get

$$\begin{aligned} &v_t - \Delta(\Psi v) + \operatorname{div}(b(x,u)v) = 0, \text{ a.e. } t \in (0,T), \\ &v(0) = 0. \end{aligned} \tag{3.3}$$

For $y = \Gamma v$, we get

$$\begin{aligned} &\frac{d}{dt}\, y - \Gamma\Delta(\Psi v) + \Gamma \operatorname{div}(b(x,u)v) = 0 \\ &y(0) = 0 \end{aligned} \tag{3.4}$$

and multiplying scalarly in L^2 with v, we get as above that

$$\begin{aligned} \frac{1}{2}\,\frac{d}{dt}\,|v(t)|^2_{-2} + \gamma_0|v(t)|_2^2 &\le |\Psi|_\infty|v(t)|_{-2}|v(t)|_2 \\ +|b|_\infty|v(t)|_2|v(t)_{-2} &\le (|\Psi|_\infty + |b|_\infty)|v(t)|_2^{\frac{3}{2}}|v(t)|_{-2}^{\frac{1}{2}}, \\ &\text{a.e. } t \in (0,T). \end{aligned} \tag{3.5}$$

This yields

$$\frac{d}{dt}\,|v(t)|^2_{-2} \le |v(t)|^2_{-2} \text{ a.e. } t \in (0,T),$$

and, therefore, $v \equiv 0$, as claimed.

4 Uniqueness in law of the McKean–Vlasov stochastic differential equations (SDEs)

Consider for $T \in (0,\infty)$ and $u_0 \in L^1 \cap L^\infty$ the McKean–Vlasov stochastic differential equation (SDE)

$$\begin{aligned} &dX(t) = b(X(t),u(t,X(t)))dt + \frac{1}{\sqrt{2}}\left(\frac{\beta(u(t,X(t)))}{u(t,X(t))}\right)^{\frac{1}{2}} dW(t), \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le t \le T, \\ &u(0,\cdot) = \xi_0, \end{aligned} \tag{4.1}$$

on $\mathbb{R}^d$. Here, $W(t),\ t \geq 0$, is an $(\mathcal{F}_t)$-Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with normal filtration $\mathcal{F}_t$, $t \geq 0$, $\xi_0 : \Omega \to \mathbb{R}^d$ is $\mathcal{F}_0$-measurable such that

$$\mathbb{P} \circ \xi_0^{-1}(dx) = u_0(x)dx,$$

and $u(t,x) = \frac{d\mathcal{L}_{X(t)}}{dx}(x)$ is the Lebesgue density of the marginal law $\mathcal{L}_{X(t)} = \mathbb{P} \circ X(t)^{-1}$ of the solution process $X(t)$, $t \geq 0$. Here, a solution process means an $(\mathcal{F}_t)$-adapted process with $\mathbb{P}$-a.s. continuous sample paths in $\mathbb{R}^d$ solving (4.1).

Theorem 4.1. *Let* $0 < T < \infty$ *and let the above conditions* (i)–(ii) *on* b *and* β *hold. Let* $X(t)$, $t \geq 0$, *and* $\widetilde{X}(t)$, $t \geq 0$, *be two solutions to* (4.1) *such that, for*

$$u(t,\cdot) := \frac{d\mathcal{L}_{X(t)}}{dx}, \quad \widetilde{u}(t,\cdot) := \frac{d\mathcal{L}_{\widetilde{X}(t)}}{dx},$$

we have

$$u, \widetilde{u} \in L^\infty((0,T) \times \mathbb{R}^d). \tag{4.2}$$

Then X *and* $\widetilde{X}$ *have the same laws, i.e.,* $\mathbb{P} \circ X^{-1} = \mathbb{P} \circ \widetilde{X}^{-1}$.

Proof. By Itô's formula, both u and $\widetilde{u}$ satisfy the (nonlinear) Fokker–Planck equation (1.1) in the sense of Schwartz distributions. Hence, by Theorem 2.1, $u = \widetilde{u}$. Furthermore, again by Itô's formula, $\mathbb{P} \circ X^{-1}$ and $\mathbb{P} \circ \widetilde{X}^{-1}$ satisfy the martingale problem with the initial condition $u_0 dx$ for the linear Komogorov operator

$$L_u := \Phi(u)\Delta + b(\cdot, u) \cdot \nabla,$$

where $\Phi(u) = \frac{\beta(u)}{u}$, $u \in \mathbb{R}$. Hence, by Theorem 3.1, the assertion follows by Lemma 2.12 in [10].

Here, for $s \in [0,T]$, the set $\mathcal{R}_{[s,T]}$, which appears in that lemma, is chosen to be the set of all narrowly continuous, probability measure-valued solutions of (3.1) having for each $t \in [s,T]$ a density $v(t,\cdot) \in L^\infty$ with respect to Lebesgue measure such that $v \in L^\infty((0,T) \times \mathbb{R}^d)$. □

Remark 4.2. We note that, by the narrow continuity, (4.2) implies that, for every $t \in [0,T]$, $u(t,\cdot), \widetilde{u}(t,\cdot) \in L^\infty$. This fact was used in the above proof.

As regards the probabilistic representation $u = \frac{\partial \mathcal{L}_X}{\partial x}$ of solutions u to the Fokker–Planck equation (1.1) via the McKean-Vlasov equation (4.1), we mention also the works [1], [3], [4], [6]. For general results involving the superposition principle, we refer to [10].

Acknowledgements. This work was supported by the DFG through CRC 1283.

References

[1] Barbu, V., *Nonlinear Differential Equations of Monotone Type in Banach Spaces*, Springer, Berlin. Heidelberg. New York, 2010.

[2] Barbu, V., Röckner, M., From nonlinear Fokker–Planck equations to solutions of distribution dependent SDE, to appear in *Ann. Probab.*, arXiv:1808.107062[math.PR].

[3] Barbu, V., Röckner, M., Probabilistic representation for solutions to nonlinear Fokker–Planck equation, *SIAM J. Math. Anal.*, 50 (4) (2018), 4246-4260.

[4] Barbu, V., Röckner, M., Solutions for nonlinear Fokker–Planck equations with measures as initial data and McKean-Vlasov equations, arXiv: 2005.02311 [math.PR].

[5] Barbu, V., Röckner, M., The evolution to equilibrium of solutions to nonlinear Fokker–Planck equations, arXiv:1904.082-91[math.PR].

[6] Barbu, V., Russo, F., Röckner, M., Probabilistic representation for solutions of an irregular porous media type equation: the irregular degenerate case, *Probab. theory Rel. Fields*, 15 (2011), 1-43.

[7] Belaribi, N., Russo, F., Uniquness for Fokker–Planck equations with measurable coefficients and applications to the fast diffusion equations, *Electron. J. Probab.*, 17 (2012), 1-28.

[8] Brezis, H., Crandall, M.G., Uniqueness of solutions of the initial-value problem for $u_t - \Delta\beta(u) = 0$, *J. Math. Pures et Appl.*, 58 (1979), 153-163.

[9] Pierre, M., Uniqueness of the solutions of $u_t - \Delta\varphi(u) = 0$ with initial data measure, *Nonlinear Analysis. Theory Methods & Applications*, 6 (2) (1982), 175-187.

[10] Trevisan, D., Well-posedness of multidimensional diffusion processes with weakly differentiable coefficients, *Electron. J. of Probab.*, Volume 21 (2016), paper no. 22, 41 pp.