OCTAV MAYER INSTITUTE OF MATHEMATICS
ROMANIAN ACADEMY, IAŞI BRANCH
Bd. Carol I nr. 10, 700506, Iaşi, Romania

PREPRINT SERIES OF THE
OCTAV MAYER INSTITUTE OF MATHEMATICS

INTERNAL CONTROLLABILITY OF PARABOLIC SYSTEMS WITH STAR AND TREE LIKE COUPLINGS

Cătălin-George Lefter
Octav Mayer Institute of Mathematics of the Romanian Academy, Iaşi
Faculty of Mathematics, University "Al. I. Cuza" Iaşi, Romania

Elena-Alexandra Melnig
Octav Mayer Institute of Mathematics of the Romanian Academy, Iaşi
Faculty of Mathematics, University "Al. I. Cuza" Iaşi, Romania

Nr. 06-2020

INTERNAL CONTROLLABILITY OF PARABOLIC SYSTEMS WITH STAR AND TREE LIKE COUPLINGS

CĂTĂLIN-GEORGE LEFTER AND ELENA-ALEXANDRA MELNIG

Abstract. We consider systems of parabolic equations coupled in zero order terms in a star-like or a tree-like shape, with an internal control acting in only one of the equations. We obtain local exact controllability to the stationary solutions of the system under hypotheses concerning the supports of the coupling coefficients. The key point is establishing appropriate Carleman estimates for the adjoint to the linearized system.

1. Introduction

In this paper we consider semilinear systems of parabolic equations coupled in zero order terms. We are interested in controllability of such systems to stationary solutions by only one control distributed in a subdomain and acting in only one of the equations. The key hypotheses insuring local controllability refer to the structure of the couplings, which describe either a star or a tree type graph, and to the support of the coupling functions or, in the linear case, to the support of the coupling coefficients.

The strategy for proving the controllability result relies on the linearization of the nonlinear system around a stationary state. The key step is obtaining the null controllability for this linear system by using an observability inequality for the adjoint system. This observability inequality is consequence of an appropriate global Carleman estimate. This in turn is obtained by combining Carleman estimates for each of the equation, but relying on diferent auxiliary functions, which are in a particular order relation, made possible by the special structure of the system. The idea of using different auxiliary functions in Carleman estimates is inspired by the work of G.Olive [14] concerning controllability of parabolic systems with controls acting in different subdomains.

Passing from the linearized system to the nonlinear one needs an L^∞ framework for the controlability of the linear system because the Carleman estimates we obtain are sensitive to zero order perturbations of the system. More regularity of the controls in the linearized problem is obtained as in the work of V.Barbu [5] (see also [6]) by using regularizing properties of the parabolic flow in a bootstrap argument. This allows an approach to the controllability of the nonlinear system by a fixed point argument, based on Kakutani theorem, as in work of J.-M. Coron, S.Guerrero and L.Rosier [6] or [4]. In fact the proof of this step follows the same lines as in [6] where the return method is used and the linearization is performed around

2010 *Mathematics Subject Classification.* 35K40, 35K58, 93B05, 93B07, 93B18.

Key words and phrases. Controllability of parabolic systems, Carleman estimates, tree-like systems, reaction-diffusion systems.

a particular trajectory, such that the linearized system is well coupled; this also is a situation where an L^∞ framework for the controllability is necessary by the same reason as in the case we are considering.

Global Carleman estimates are by now a classical tool in proving observability inequalities, and they were established in the context of controllability for parabolic equation O.Yu.Imanuvilov (see O.Yu.Imanuvilov and A.Fursikov [9]). Since then this type of estimates was extensively developed, refined and used in other areas such as stabilization or inverse problems.

The study of controlled systems of parabolic equations, with fewer controls than equations, needs appropriate Carleman estimates for the adjoint system. These estimates usually involve partial observations. We recall the study of cascade like systems of parabolic equations, with one control and space depending couplings, in the paper of Luz de Teresa and M.Gonzáles-Burgos [10]. In the case of zero order couplings with constant or time dependent coupling coefficients there is an extended interest on obtaining algebraic conditions of Kalman type for controllability; in this direction we cite the papers of F.Ammar-Khodja, A.Benabdallah, C.Dupaix and M.Gonzáles-Burgos [2, 1] or the work of F.Ammar-Khodja, F.Chouly and M. Duprez [3]. Observability estimates for linear systems (not only parabolic) coupled with constant coupling coefficients in the dominant part and/or in the zero order terms were established by E.Zuazua and P.Lissy [13]; such estimates are established under Kalman rank conditions satisfied by the pair of the coupling and control matrices.

2. Preliminaries and statement of the problem

Let $\Omega \subset \mathbb{R}^N$ be a bounded connected domain with a C^2 boundary $\partial\Omega$ and let $\omega_0 \subset\subset \Omega$. Let $T > 0$ and denote by $Q = (0,T) \times \Omega$ and for $\omega \subset \Omega$ write $Q_\omega = (0,T) \times \omega$.

We consider systems of $(n+1)$ parabolic equations coupled in zero order terms through nonlinear functions, with one internally distributed control, acting in ω_0 and entering only the first equation. The main goal is obtaining local exact controllability to some stationary solution for the nonlinear system.

In the first part of the paper we study systems of parabolic equations with star-like couplings which refer to the sistuation where y_k is actuated in the corresponding parabolic equation through a nonliniarity depending only on y^0, y^k. Such a star-like coupled system has the form:

$$\begin{cases} D_t y_0 - \Delta y_0 = g_0(x) + f_0(x, y_0) + \chi_{\omega_0} u, & \text{in } (0,T) \times \Omega, \\ D_t y_i - \Delta y_i = g_i(x) + f_i(x, y_0, y_i),\ i \in \overline{1,n}, & \text{in } (0,T) \times \Omega, \\ y_0 = ... = y_n = 0, & \text{on } (0,T) \times \partial\Omega, \\ y(0,\cdot) = y^0, \end{cases} \tag{2.1}$$

where $g_j \in L^\infty(\Omega)$, $j \in \overline{0,n}$. We denote by $\chi_{\omega_0} v$ the extension of $v : \omega_0 \to \mathbb{R}$ with 0 to the whole domain Ω. The control function is $u : [0,T] \times \omega_0 \longrightarrow \mathbb{R}$, acting in the equation of y_0 and acting on the other components of the solution, $y_1, ..., y_n$, through the corresponding coupling terms containing y_0.

Consider a stationary state $\overline{y}=(\overline{y}_0,...,\overline{y}_n), \overline{y}_j\in L^\infty(\Omega), j\in\overline{0,n}$, solution to the elliptic system:

$$\left\{\begin{array}{ll} -\Delta\overline{y}_0=g_0(x)+f_0(x,\overline{y}_0), & x\in\Omega,\\ -\Delta\overline{y}_i=g_i(x)+f_i(x,\overline{y}_0,\overline{y}_i),\ i\in\overline{1,n}, & x\in\Omega,\\ \overline{y}_0=...=\overline{y}_n=0, & x\in\partial\Omega. \end{array}\right. \tag{2.2}$$

Observe in fact that, by elliptic regularity, an L^∞ stationary solution is a smooth solution.

Concerning the coupling terms we assume the following hypotheses:

(H1) $f_0:\mathbb{R}^N\times\mathbb{R}\longrightarrow\mathbb{R}$, $f_i:\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}, i\in\overline{1,n}$ are C^1 functions and there exist $\omega_1,...\omega_n\subset\Omega$, open nonempty subsets of Ω such that

$$(\omega_i\cap\omega_0)\setminus\bigcup_{j\neq 0,i}\omega_j\neq\emptyset,\ \forall i\in\overline{1,n}, \tag{2.3}$$

and for all $i\in\overline{1,n}$ we have

$$f_i(x,y_0,y_i)=0\,\forall x\in\Omega\setminus\omega_i,\ y_0,y_i\in\mathbb{R}; \tag{2.4}$$

(H2) The following coupling condition holds:

$$\text{supp}\frac{\partial f_i}{\partial y_0}(x,\overline{y}_0(x),\overline{y}_i(x))\cap\left\{(\omega_i\cap\omega_0)\setminus\overline{\bigcup_{j\neq 0,i}\omega_j}\right\}\neq\emptyset, \tag{2.5}$$

We consider first a controlled linear system which will appear through a linearization procedure:

$$\left\{\begin{array}{ll} D_tz_0-\Delta z_0=c_0(t,x)z_0+\chi_{\omega_0}u, & (0,T)\times\Omega,\\ D_tz_i-\Delta z_i=a_{i0}(t,x)z_0+c_i(t,x)z_i,\ i\in\overline{1,n}, & (0,T)\times\Omega,\\ z_0=...=z_n=0, & (0,T)\times\partial\Omega, \end{array}\right. \tag{2.6}$$

For $M,\delta>0$, and open subsets $\underline{\omega_i}\subset\subset(\omega_i\cap\omega_0)\setminus\bigcup_{j\neq 0,i}\omega_j$ we introduce the following classes of coefficients sets:

$$\begin{aligned} &\mathcal{E}_{M,\delta,\{\underline{\omega_i}\}_i}=\Big\{E=\{a_{i0},c_j\}_{i\in\overline{1,n},j\in\overline{0,n}}:a_{i0},c_j\in L^\infty(Q),\\ &\|a_{i0}\|_{L^\infty},\|c_j\|_{L^\infty}\le M, a_{i0}=0\text{ in }Q\setminus Q_{\omega_i},\text{ and }|a_{i0}|\ge\delta\text{ on }Q_{\underline{\omega_i}}\Big\}. \end{aligned} \tag{2.7}$$

We prove first that such linear systems with coefficients in $\mathcal{E}_{M,\delta,\{\underline{\omega_i}\}_i}$ are null controllable with norm L^2 and L^∞ of the control uniformly bounded by a constant $C=C(M,\delta,\{\underline{\omega_i}\}_i)$.

In order to achieve this goal we consider the adjoint system:

$$\left\{\begin{array}{ll} -D_tp_0-\Delta p_0=c_0(t,x)p_0+\sum_{i=1}^n a_{i0}(t,x)p_i, & (0,T)\times\Omega,\\ -D_tp_i-\Delta p_i=c_i(t,x)p_i,\ i\in\overline{1,n}, & (0,T)\times\Omega,\\ p_0=...=p_n=0, & (0,T)\times\partial\Omega. \end{array}\right. \tag{2.8}$$

and we prove an observability inequality as consequence of an appropriate Carleman estimate. The Carleman estimate we establish in the next section gives us more than just observability, it helps obtaining a priori estimates for the control driving the solution of the linear system to zero and, as the constants appearing in the Caleman estimates are depending only on $M,\delta,\{\underline{\omega_i}\}_i$, the estimates on the control will result uniform. This fact is essential in the fixed point argument when dealing with the nonlinear system.

In order to reformulate the problem in an abstract functional framework let the state space be the Hilbert space $H = [L^2(\Omega)]^{n+1}$ and the control space $U = L^2(\omega_0)$. Consider the operator

$$\mathbf{A} : D(\mathbf{A}) \subset H \longrightarrow H, D(\mathbf{A}) = (H_0^1(\Omega) \cap H^2(\Omega))^{n+1}, \mathbf{A}z = \Delta z,$$

and the control operator

$$\mathbf{B} : U \to H,\ \mathbf{B}u = \chi_{\omega_0} B u,\ B = (1, 0, \dots, 0)^\top.$$

Then, problem (2.1) may be written in abstract form:

$$\begin{cases} D_t y = \mathbf{A}y + \mathbf{f}(y) + \mathbf{B}u, \quad t > 0, \\ y(0) = y^0. \end{cases} \tag{2.9}$$

where $\mathbf{f}(y) = f(\cdot, y(\cdot))$. The linear problem (2.6) may be reformulated as:

$$\begin{cases} D_t z = \mathbf{A}z + \mathbf{A_0}(t)z + \mathbf{C}(t)z + \mathbf{B}u, \quad t > 0, \\ z(0) = z^0, \end{cases} \tag{2.10}$$

where $\mathbf{C}(t)z = C_0(t,\cdot)z(\cdot)$ and $\mathbf{A_0}(t)z = A_0(t,\cdot)z(\cdot)$ where $C_0(t,x)$ is the diagonal matrix $C_0(t,x) = diag(c_i(t,x))_{i=\overline{0,n}}$ and the coupling matrix $A_0(t,x)$ has only one nonzero column, that is the first one and is given by

$$A_0(t,x) = (0, a_{10}, \dots, a_{n0})^\top \cdot (1, 0, \dots, 0).$$

For simplicity, when there is no confusion, we denote the norms of functions $z \in [L^2(\Omega)]^{n+1}$, or $z \in [H^1(\Omega)]^{n+1}$ *etc.* as $\|z\|_{L^2(\Omega)}$, respectively $\|z\|_{H^1(\Omega)}$ *etc.*.

Null controllabity for the linear system (2.10) above is equivalent to an observability inequality

$$\|p(0)\|^2_{L^2(\Omega)} \le C(M,\delta) \int_0^T \|\mathbf{B}^* p\|^2_{L^2(\omega_0)} dt, \qquad \text{for some } C(M,\delta) > 0, \tag{2.11}$$

for all solutions p to the adjoint equation

$$-p' = \mathbf{A}p + \mathbf{A_0^*}p + \mathbf{C}p \tag{2.12}$$

where $\mathbf{A_0^*}p = A_0^\top p, \mathbf{B}^*p = B^\top p|_{\omega_0}$.

We extend our study to parabolic systems with tree-like couplings. In fact we will treat only linear equations with appropriate hypotheses for the coupling coefficients in a tree-like structure. Passing from linear results of controllability to local controllability for nonlinear systems may be obtaiend by exactly the same procedure as in the star-like case. An example of linear parabolic system with tree-like couplings is the following:

$$\begin{cases} D_t z_0 - \Delta z_0 = c_0(t,x)z_0 + \chi_{\omega_0}u, & \text{in } (0,T)\times\Omega, \\ D_t z_1 - \Delta z_1 = a_{10}(t,x)z_0 + c_1(t,x)z_1, & \text{in } (0,T)\times\Omega, \\ D_t z_2 - \Delta z_2 = a_{20}(t,x)z_0 + c_2(t,x)z_2, & \text{in } (0,T)\times\Omega, \\ D_t z_3 - \Delta z_3 = a_{31}(t,x)z_1 + c_3(t,x)z_3, & \text{in } (0,T)\times\Omega, \\ D_t z_4 - \Delta z_4 = a_{41}(t,x)z_1 + c_2(t,x)z_4, & \text{in } (0,T)\times\Omega, \\ z_0 = \dots = z_4 = 0, & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{2.13}$$

and the general form of system with tree like couplings will be discussed in §6.

The paper is organized as follows:

- In §3 we prove appropriate Carleman estimates for adjoint system (2.8) in either $L^2 - L^2$ or $L^\infty - L^2$ settings. This will be Theorem 1
- In §4 we prove the null controllability of linear system (2.6). The approach uses a family of optimal control problems with penalized final cost. One then obtains besides controllability an estimate for the control in both L^2 and L^∞ norms by using the previous Carleman estimates. This is Theorem 2.
- §5 is devoted to the local controllability in L^∞ of nonlinear system (2.1). The fact that controllability has to be proved in L^∞ is due to the high sensitivity of the Carleman estimates with respect to the coupling coefficients, which is not the case when controls act in each equation of the system. The argument is similar to that used in [6].
- In §6 we extend results of controllability, with one distributed scalar control, for linear systems of parabolic equations, of the form (2.13), with tree-like couplings. The key point here is obtaining appropriate Carleman estimates. Local controllability for nonlinear systems with tree-like couplings is also discussed.

3. Carleman estimates and observability

In this section we establish an L^2 Carleman estimate that will help proving an observability inequality fot the adjoint problem (2.8). This L^2 Carleman inequality and parabolic regularity are the starting point in obtaining an L^∞ control through a bootstrap argument.

We recall the classical Carleman estimate for a generic nonhomogeneous parabolic problem,

$$\begin{cases} D_t p + Lp = h, & \text{in } (0,T)\times\Omega, \\ p = 0, & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{3.1}$$

where L is an uniformly elliptic operator of second order. Denote by $Q := (0,T)\times\Omega$ and, for $\omega \subset\subset \Omega$, $Q_\omega := (0,T)\times\omega$. The solution is observed in Q_ω for sources $h \in L^2(Q)$.

We introduce the function

$$\psi \in C^2(\overline{\Omega}),\ \psi|_{\partial\Omega} = k > 0,\ k < \psi < \frac{3}{2}k \text{ in } \Omega, \{x \in \overline{\Omega} : |\nabla\psi(x)| = 0\} \subset\subset \omega,$$

and the weight functions

$$\varphi(t,x) := \frac{e^{\lambda\psi(x)}}{t(T-t)}, \quad \alpha(t,x) := \frac{e^{\lambda\psi(x)} - e^{1.5\lambda\|\psi\|_{C(\overline{\Omega})}}}{t(T-t)}. \tag{3.2}$$

Then, the classical global Carleman estimate (see [9], [8]) is the following:

Lemma 1. *There exist λ_0, s_0 and $C > 0$ such that if $\lambda > \lambda_0, s \geq s_0$, the following inequality holds:*

$$\begin{aligned} &\int_Q \left[(s\varphi)^{-1}(|D_t p|^2 + |D^2 p|) + s\lambda^2\varphi|Dp|^2 + s^3\lambda^4\varphi^3|p|^2\right] e^{2s\alpha} dxdt \\ &\leq C\int_{Q_\omega} s^3\lambda^4\varphi^3|p|^2 e^{2s\alpha} dxdt + \int_Q |h|^2 e^{2s\alpha} dxdt \end{aligned} \tag{3.3}$$

for all $p \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))$ solution of (3.1).

We establish a Carleman estimate for the adjoint problem with source $g \in L^2(Q)^{n+1}$ and observations on the subdomain ω_0. The adjoint system to problem (2.6) is

$$(3.4)\quad \begin{cases} -D_t p_0 - \Delta p_0 = c_0(t,x)p_0 + \sum_{i=1}^n a_{i0}(t,x)p_i + g_0, & (0,T)\times\Omega, \\ -D_t p_i - \Delta p_i = c_i(t,x)p_i + g_i,\ i \in \overline{1,n}, & (0,T)\times\Omega, \\ p_0 = \dots = p_n = 0, & (0,T)\times\partial\Omega. \end{cases}$$

In the following we are going to write Carleman estimates for each equation in (2.8) by using in each case corresponding subdomains of observation and appropriately chosen weight functions. We proceed as follows:

Consider open subsets

$$\tilde{\omega}_j \subset\subset \underline{\omega}_j$$

and denote as above by $Q_{\tilde{\omega}_j} = (0,T)\times\tilde{\omega}_j$; take the auxiliary functions $\psi_j, j = \overline{0,n}$, with the following properties (where we have denoted by $\tilde{\omega}_0 := \omega_0$):

$$(3.5)\qquad \psi_j := \eta_j + K_j,\ j \in \overline{0,n},$$

$$\eta_j \in C^2(\overline{\Omega}),\ 0 < \eta_j \text{ in } \Omega, \quad \eta_j|_{\partial\Omega} = 0, \quad \{x \in \overline{\Omega} : |\nabla\eta_j(x)| = 0\} \subset\subset \tilde{\omega}_j,$$

for some fixed positive constants $K_j > 0$ such that

$$(3.6)\qquad \psi_i > \psi_0 \text{ in } \Omega$$

and

$$(3.7)\qquad \frac{\sup\psi_j}{\inf\psi_j} < \frac{8}{7}, \forall j \in \overline{0,n}.$$

Let $0 < \epsilon < \inf\psi_i, i \in \overline{0,n}$ a small positive number and denote by

$$(3.8)\qquad \overline{\psi} = \sup_{x\in\Omega}\sup_{j\in\overline{0,n}}\psi_j(x) + \epsilon, \qquad \underline{\psi} = \inf_{x\in\Omega}\inf_{j\in\overline{0,n}}\psi_j(x) - \epsilon.$$

Introduce also, for parameters $s, \lambda > 0$ the auxiliary functions:

$$(3.9)\qquad \varphi_j(t,x) := \frac{e^{\lambda\psi_j(x)}}{t(T-t)}, \quad \alpha_j(t,x) := \frac{e^{\lambda\psi_j(x)} - e^{1.5\lambda\overline{\psi}}}{t(T-t)}, \forall j \in \overline{0,n}$$

and

$$(3.10)\qquad \overline{\varphi}(t) = \overline{\varphi}^\lambda(t) := \frac{e^{\lambda\overline{\psi}}}{t(T-t)}, \quad \overline{\alpha}(t) = \overline{\alpha}^\lambda(t) := \frac{e^{\lambda\overline{\psi}} - e^{1.5\lambda\overline{\psi}}}{t(T-t)},$$

$$(3.11)\qquad \underline{\varphi}(t) = \underline{\varphi}^\lambda(t) := \frac{e^{\lambda\underline{\psi}}}{t(T-t)}, \quad \underline{\alpha}(t) = \underline{\alpha}^\lambda(t) := \frac{e^{\lambda\underline{\psi}} - e^{1.5\lambda\overline{\psi}}}{t(T-t)}.$$

Remark 1. (i) *As we are going to compare the various Carleman estimates stated for each equation of the linear adjoint system, we will need to compare the weights which are involved in thgose inequalities. For this purpose let us observe that given $m_0 > 0$ there exist $s_0 = s_0(m_0), \lambda_0 = \lambda_0(m_0) > 0$ such that for all $s > s_0, \lambda > \lambda_0$, $|m| \le m_0$ and $t \in (0,T)$, the following inequality holds:*

$$(3.12)\qquad e^{s\underline{\alpha}} \le s^m\varphi_i^m e^{s\alpha_i} \le e^{s\overline{\alpha}},$$

$$e^{s\alpha_0} \le s^m \varphi_i^m e^{s\alpha_i}. \tag{3.13}$$

(ii) *Observe that if in (3.5) we replace K_i with $K_i + M$ with the constant $M > 0$ big enough, the above properties of the auxiliary functions remain valid and, moreover, we may assume that*

$$\frac{\overline{\psi}}{\underline{\psi}} \le \frac{3}{2}. \tag{3.14}$$

This extra assumption implies that there exist $\bar{s}_0 > 0, \bar{\lambda}_0 > 0$ such that if $s > \bar{s}_0$ $\lambda > \bar{\lambda}_0$,

$$|D_t\varphi_i| \le C\varphi_i^2, \quad |D_t\alpha_i| \le C\varphi_i^2, \quad |D_t^2\alpha_i| \le C\varphi_i^3. \tag{3.15}$$

(iii) *Observe that for λ big enough, say $\lambda > \overline{\lambda}$, we have*

$$\frac{\underline{\underline{\alpha^\lambda}}}{\overline{\alpha^\lambda}} < 2. \tag{3.16}$$

Indeed, this is a consequence to the fact that $\lim_{\lambda\to+\infty} \frac{\underline{\underline{\alpha^\lambda}}}{\overline{\alpha^\lambda}} = 1$, uniformly with respect to $(t,x) \in Q$.

In this section we prove the following Carleman estimate which has as consequence the appropriate observability inequality for the adjoint system (3.1).

Theorem 1. *There exist constants λ_0, s_0 such that for $\lambda > \lambda_0$ there exists a constant $C > 0$ depending on $(M, \delta, \{\underline{\omega}_i\}_i, \lambda)$, such that, for any $s \ge s_0$, the following inequality holds:*

$$\begin{aligned} &\int_Q (|D_tp|^2 + |D^2p|^2 + |Dp|^2 + |p|^2) e^{2s\underline{\alpha}} dxdt \\ &\le C \int_{Q_{\omega_0}} |p_0|^2 e^{2s\overline{\alpha}} dxdt + C\int_Q |g|^2 e^{2s\overline{\alpha}} dxdt \end{aligned} \tag{3.17}$$

for all $p \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))$ solution of (3.4).

Moreover, there exist $m_0 \in \mathbb{N}$ and $\delta_1 > 0$ such that for the homogeneous adjoint system (i.e. taking $g \equiv 0$), we have the following $L^\infty - L^2$ Carleman estimate

$$\|pe^{(s+m_0\delta_1)\underline{\alpha}}\|_{L^\infty(Q)} \le C\|p_0 e^{s\overline{\alpha}}\|_{L^2(Q_{\omega_0})}. \tag{3.18}$$

Proof. The second remark above is useful when obtaining Carleman estimates, since the weights here are slightly different with respect to those used in (see [9]) or [7]. However, this remark allows following the same lines of proof and we may write Carleman estimate (3.3) for each equation $j \in \overline{0,n}$ with observation domain $\tilde{\omega}_j$ and auxiliary functions and weight functions $\psi_j, \varphi_j, \alpha_j$. Thus, there exist $s_0 > 0, C > 0$ such that for any $s \ge s_0$, the following inequalities hold:

(1) For p_0 we have

$$\begin{aligned}
&\int_Q \left[(s\varphi_0)^{-1}(|D_tp_0|^2+|D^2p_0|^2)+s\varphi_0|Dp_0|^2+s^3\varphi_0^3|p_0|^2\right]e^{2s\alpha_0}dxdt\\
&\le C\left[\int_{Q_{\omega_0}} s^3\varphi_0^3|p_0|^2e^{2s\alpha_0}dxdt+\int_Q\left|\sum_{i=1}^n a_{i0}p_i+g_0\right|^2e^{2s\alpha_0}dxdt\right]\le\\
&\le C\left[\int_{Q_{\omega_0}} s^3\varphi_0^3|p_0|^2e^{2s\alpha_0}dxdt\right.\\
&\left.+n^2M^2\sum_{i=1}^n\int_Q|p_i|^2e^{2s\alpha_i}dxdt+\int_Q|g_0|^2e^{2s\alpha_i}dxdt\right].
\end{aligned}\tag{3.19}$$

(2) For $p_i, i\in\overline{1,n}$ we have:

$$\begin{aligned}
&\int_Q\left[(s\varphi_i)^{-1}(|D_tp_i|^2+|D^2p_i|^2)+s\varphi_i|Dp_i|^2+s^3\varphi_i^3|p_i|^2\right]e^{2s\alpha_i}dxdt\\
&\le C\int_{Q_{\tilde\omega_i}} s^3\varphi_i^3|p_i|^2e^{2s\alpha_i}dxdt+C\int_Q|g_i|^2e^{2s\alpha_i}dxdt.
\end{aligned}\tag{3.20}$$

We sum the above Carleman inequalities we obtain for some constant $C=C(M,\{\omega_j\}_j)>0$ that

$$\begin{aligned}
&\sum_{j=0}^n\left\{\int_Q\left[(s\varphi_j)^{-1}(|D_tp_j|^2+|D^2p_j|^2)+s\varphi_j|Dp_j|^2+s^3\varphi_j^3|p_j|^2\right]e^{2s\alpha_j}dxdt\right\}\\
&\le C\left[\int_{Q_{\omega_0}} s^3\varphi_0^3|p_0|^2e^{2s\alpha_0}dxdt+\sum_{i=1}^n\left(\int_{Q_{\tilde\omega_i}} s^3\varphi_i^3|p_i|^2e^{2s\alpha_i}dxdt\right)\right.\\
&\left.+\sum_{j=0}^n\left(\int_{Q_{\tilde\omega_j}}|g_j|^2e^{2s\alpha_j}dxdt\right)\right].
\end{aligned}\tag{3.21}$$

At this point we have to properly estimate the terms containing p_i on $\tilde\omega_i, i\in\overline{1,n}$ from the right hand-side in terms of the component p_0 observed on $\tilde\omega_0$. For this purpose we will use the first equation of (2.8) considered on $\omega_i\cap\omega_0$, which by hypothesis (2.7) is coupled only to p_i:

$$D_tp_0+\Delta p_0+c_0p_0+a_{i0}p_i=g_0 \text{ in } (0,T)\times\omega_i\cap\omega_0.\tag{3.22}$$

Consider the cutoff functions $\gamma_i, i\in\overline{1,n}$ with the properties

$$\begin{aligned}
&\gamma_i\in C_0^\infty(\omega_i),\ |\gamma_i|\le 1, \text{supp }\gamma_i=\overline{\underline{\omega}_i}\\
&\gamma_i= \text{ sign }(a_{i_0}|_{\underline{\omega}_i}) \text{ on } \tilde\omega_i, \gamma_i\neq 0 \text{ in } \underline{\omega}_i.
\end{aligned}$$

where sign (a_{i_0}) is the sign of a_{i0} in $\underline{\omega}_i$, which, by hypothesis (2.7) and continuity is nonzero and constant in $\tilde\omega_i$. Multiply, scalarly in $L^2(Q_{\omega_0})$, the equation (3.22) by $\gamma_is^3\varphi_i^3p_ie^{2s\alpha_i}$:

$$\begin{aligned}
&\int_{Q_{\underline{\omega}_i}}\gamma_ia_{i0}(x)s^3\varphi_i^3|p_i|^2e^{2s\alpha_i}dxdt\\
&=\int_{Q_{\underline{\omega}_i}}\gamma_is^3\varphi_i^3(-c_0p_0-D_tp_0-\Delta p_0-g_0)p_ie^{2s\alpha_i}dxdt
\end{aligned}\tag{3.23}$$

We use (2.7) to say that that there exists a constant such that

$$\delta \int_{Q_{\tilde{\omega}_i}} s^3 \varphi_i^3 |p_i|^2 e^{2s\alpha_i} dxdt \leq \int_{Q_{\tilde{\omega}_i}} |a_{i0}(x)| s^3 \varphi_i^3 |p_i|^2 e^{2s\alpha_i} dxdt \leq \int_{Q_{\underline{\omega}_i}} a_{i0}(x) s^3 \varphi_i^3 |p_i|^2 e^{2s\alpha_i} dxdt. \tag{3.24}$$

We estimate each term from the right hand-side of (3.23) using the properties of $\gamma_j, j \in \overline{0,n}$. Let $C > 0$ denoting various constants depending on δ, M and $\underline{\omega}_i, \tilde{\omega}_i$.

For the first term in right side of (3.23) we have:

$$\begin{aligned} &\left| \int_{Q_{\underline{\omega}_i}} \gamma_i s^3 \varphi_j^3 (-c_0 p_0) p_i e^{2s\alpha_i} dxdt \right| \\ &\leq M \left(\int_{Q_{\underline{\omega}_i}} s^2 \varphi_i^2 |p_i|^2 e^{2s\alpha_i} dxdt \right)^{\frac{1}{2}} \left(\int_{Q_{\underline{\omega}_i}} s^4 \varphi_i^4 |p_0|^2 e^{2s\alpha_i} dxdt \right)^{\frac{1}{2}} \\ &\leq \int_{Q_{\underline{\omega}_i}} s^2 \varphi_i^2 |p_i|^2 e^{2s\alpha_i} dxdt + M^2 \int_{Q_{\underline{\omega}_i}} s^4 \varphi_i^4 |p_0|^2 e^{2s\alpha_i} dxdt. \end{aligned} \tag{3.25}$$

The same computation gives an estimate for the term involving the source:

$$\begin{aligned} &\left| \int_{Q_{\underline{\omega}_i}} \gamma_i s^3 \varphi_j^3 (-g_0) p_i e^{2s\alpha_i} dxdt \right| \\ &\leq \int_{Q_{\underline{\omega}_i}} s^2 \varphi_i^2 |p_i|^2 e^{2s\alpha_i} dxdt + M^2 \int_{Q_{\underline{\omega}_i}} s^4 \varphi_i^4 |g_0|^2 e^{2s\alpha_i} dxdt. \end{aligned} \tag{3.26}$$

Observe now that we have the following estimates for the weight functions, with a constant cst not depending on s:

$$|\gamma_i s^3 D_t(e^{2s\alpha_i} \varphi_i^3)| = |\gamma_i s^3 (e^{2s\alpha_i} 2s D_t \alpha_i \varphi_i^3 + 3 e^{2s\alpha_i} \varphi^2 D_t \varphi_i)| \leq cst\, e^{2s\alpha_i} s^5 \varphi_i^5 \tag{3.27}$$

and

$$|s^3 \Delta(\gamma_i \varphi_i^3 p_i e^{2s\alpha_i})| \leq cst\, s^3 \varphi_i^3 (s^2 \varphi_i^2 |p_i| + s\varphi_i |\nabla p_i| + |\Delta p_i|) e^{2s\alpha_i}. \tag{3.28}$$

We now proceed with estimating the second term in (3.23) using, as usually in Carleman estimates, integration by parts:

$$\begin{aligned} &\left| \int_{Q_{\underline{\omega}_i}} \gamma_i s^3 \varphi_i^3 (-D_t p_0) p_i e^{2s\alpha_i} dxdt \right| = \left| \int_{Q_{\underline{\omega}_i}} s^3 D_t(\varphi_i^3 p_i e^{2s\alpha_i}) p_0 dxdt \right| \\ &\leq \left| \int_{Q_{\underline{\omega}_i}} s^3 D_t(\varphi_i^3 e^{2s\alpha_i}) p_i p_0 dxdt \right| + \left| \int_{Q_{\underline{\omega}_i}} s^3 \varphi_i^3 e^{2s\alpha_i} D_t p_i p_0 dxdt \right| \\ &\leq C \left| \int_{Q_{\underline{\omega}_i}} e^{2s\alpha_i} s^5 \varphi_i^5 p_j p_0 dxdt \right| + \left| \int_{Q_{\underline{\omega}_i}} e^{2s\alpha_i} s^3 \varphi_i^3 D_t p_i p_0 dxdt \right| \\ &\leq \int_{Q_{\underline{\omega}_i}} s^2 \varphi_i^2 |p_i|^2 e^{2s\alpha_i} dxdt + C \int_{Q_{\underline{\omega}_i}} s^8 \varphi_i^8 |p_0|^2 e^{2s\alpha_i} dxdt \\ &+ \int_{Q_{\underline{\omega}_i}} (s\varphi)^{-2} |D_t p_i|^2 e^{2s\alpha_i} dxdt + C \int_{Q_{\underline{\omega}_i}} s^8 \varphi_i^8 |p_0|^2 e^{2s\alpha_i} dxdt. \end{aligned} \tag{3.29}$$

We proceed now with estimating the third term in right hand side of (3.23):

$$
\begin{aligned}
&\left|\int_{Q_{\underline{\omega}_i}} \gamma_i s^3\varphi_i^3(-\Delta p_0)p_i e^{2s\alpha_i}dxdt\right| = \left|\int_{Q_{\underline{\omega}_i}} s^3\Delta(\gamma_i\varphi_i^3 p_i e^{2s\alpha_i})p_0 dxdt\right| \\
&\leq C\int_{Q_{\underline{\omega}_i}} s^3\varphi_i^3(s^2\varphi_i^2|p_i| + s\varphi_i|\nabla p_i| + |\Delta p_i|)e^{2s\alpha_i}|p_0|dxdt \\
&\leq \int_{Q_{\underline{\omega}_i}} [s^2\varphi_i^2|p_i|^2 + |\nabla p_i|^2 + (s\varphi_i)^{-2}|\Delta p_i|^2]e^{2s\alpha_i}dxdt \\
&+ C\int_{Q_{\underline{\omega}_i}} s^8\varphi_i^8|p_0|^2e^{2s\alpha_i}dxdt.
\end{aligned}
\tag{3.30}
$$

Using (3.25),(3.26),(3.29), (3.30) and (3.24) we have, for $i \in \overline{1,n}$ that

$$
\begin{aligned}
&\int_{Q_{\tilde{\omega}_i}} s^3\varphi_i^3|p_i|^2e^{2s\alpha_i}dxdt \leq C\int_{Q_{\underline{\omega}_i}} s^8\varphi_i^8|p_0|^2e^{2s\alpha_i}dxdt \\
&+ \int_{Q_{\underline{\omega}_i}} \left[(s\varphi_i)^{-2}(|\Delta p_i|^2 + |D_t p_i|^2) + s^2\varphi_i^2|p_i| + |\nabla p_i|^2\right]e^{2s\alpha_i}dxdt \\
&+ C\sum_{i=1}^n \int_{Q_{\underline{\omega}_i}} s^4\varphi_0^4|g_0|^2e^{2s\alpha_i}dxdt.
\end{aligned}
\tag{3.31}
$$

Going back to (3.21), we have

$$
\begin{aligned}
&\sum_{j=0}^n \left\{\int_Q \left[(s\varphi_j)^{-1}(|D_t p_j|^2 + |D^2 p_j|^2) + s\varphi_j|Dp_j|^2 + s^3\varphi_j^3|p_j|^2\right]e^{2s\alpha_j}dxdt\right\} \\
&\leq C\int_{Q_{\omega_0}} s^3\varphi_0^3|p_0|^2e^{2s\alpha_0}dxdt + C\sum_{i=1}^n\Bigg(\int_{Q_{\underline{\omega}_i}} s^8\varphi_i^8|p_0|^2e^{2s\alpha_i}dxdt \\
&+ \int_{Q_{\underline{\omega}_i}} \left[(s\varphi_i)^{-2}(|D^2 p_i|^2 + |D_t p_i|^2) + s^2\varphi_i^2|p_i|^2 + |Dp_i|^2\right]e^{2s\alpha_i}dxdt\Bigg) \\
&+ C\sum_{i=1}^n\int_{Q_{\underline{\omega}_i}} s^4\varphi_0^4|g_0|^2e^{2s\alpha_i}dxdt + C\sum_{j=0}^n\int_Q |g_j|^2e^{2s\alpha_j}dxdt.
\end{aligned}
\tag{3.32}
$$

We now absorb the integral terms containing p_i in the right hand side into the corresponding higher order terms in the left side of the above inequality, by increasing s and taking it big enough. We obtain:

$$
\begin{aligned}
&\sum_{j=0}^n \left\{\int_Q \left[(s\varphi_j)^{-1}(|D_t p_j|^2 + |D^2 p_j|^2) + s\varphi_j|Dp_j|^2 + s^3\varphi_j^3|p_j|^2\right]e^{2s\alpha_j}dxdt\right\} \\
&\leq C\int_{Q_{\omega_0}} s^3\varphi_0^3|p_0|^2e^{2s\alpha_0}dxdt + C\sum_{i=1}^n\int_{Q_{\underline{\omega}_i}} s^8\varphi_i^8|p_0|^2e^{2s\alpha_i}dxdt. \\
&+ C\sum_{i=1}^n\int_{Q_{\underline{\omega}_i}} s^4\varphi_0^4|g_0|^2e^{2s\alpha_i}dxdt + C\sum_{j=0}^n\int_Q |g_j|^2e^{2s\alpha_j}dxdt.
\end{aligned}
\tag{3.33}
$$

Now if we use Remark 1 in order to take a smaller weight in the left side and a greater one in the right side. Then there exist $s_0 > 0$ and $C = C(M, \delta, \{\underline{\omega}_j\}_j)$ such that the following Carleman estimate is true for all $s \geq s_0$:

$$\begin{aligned}(3.34)\qquad &\sum_{j=0}^{n}\left[\int_Q \left(|D_t p_j|^2 + |D^2 p_j|^2 + |Dp_j|^2 + |p_j|^2\right) e^{2s\underline{\alpha}} dxdt\right] \\ &\leq C\int_{Q_{\omega_0}} |p_0|^2 e^{2s\overline{\alpha}} dxdt + C\int_Q |g|^2 e^{2s\overline{\alpha}} dxdt.\end{aligned}$$

□

Concerning the $L^\infty - L^2$ Carleman estimate for the solution of the adjoint problem (2.8) we proceed in the same way as is in [5, 6] or[12]. We need to use the maximal regularity result in L^p spaces for parabaolic problems (see [11]) and Sobolev embeddings for anisotropic Sobolev spaces which are contained in the following lemma:

Lemma 2 ([11], Lemma 3.3)**.** *Let $z \in W_r^{2,1}(Q)$.*
Then $z \in Z_1$ where

$$Z_1 = \begin{cases} L^s(Q) & \text{with } s \leq \frac{(N+2)r}{N+2-2r} \quad \text{when } r < \frac{N+2}{2} \\ L^s(Q) & \text{with } s \in [1, \infty[, \quad \text{when } r = \frac{N+2}{2} \\ C^{\alpha,\alpha/2}(Q) & \text{with } 0 < \alpha < 2 - \frac{N+2}{r}, \quad \text{when } r > \frac{N+2}{2}\end{cases}$$

and there exists $C = C(Q, p, N)$ such that

$$\|z\|_{Z_1} \leq C\|z\|_{W_r^{2,1}(Q)}.$$

Using the above regularity result we consider the following sequence of numbers:

$$(3.35)\qquad \sigma_0 = 2, \quad \sigma_j := \begin{cases} \dfrac{(N+2)\sigma_{j-1}}{N+2-2\sigma_{j-1}}, \text{ if } \sigma_{j-1} < \frac{N+2}{2}, \\ \frac{3}{2}\sigma_{j-1}, \text{ if } \sigma_{j-1} \geq \frac{N+2}{2}, \end{cases}$$

such that by Lemma 2 we have

$$W_{\sigma_{m-1}}^{2,1}(Q) \subset L^{\sigma_m}(Q).$$

Now, let us fix a $\delta_1 > 0$ and a sequence $(q^j)_{j>0}$ defined by

$$q^j := p^\varepsilon e^{(s+j\delta_1)\underline{\alpha}}.$$

Then $q^j = (q_0^j, \dots, q_n^j)^\top$ is solution to the problem

$$\begin{aligned}(3.36)\qquad &D_t q^j + \mathbf{A}q^j + Cq^j + A_0^\top q^j = (s + j\delta_1)D_t\underline{\alpha} q^j, \\ &q^j(T) = 0.\end{aligned}$$

Observe that the right-hand side may be bounded in terms of q^{j-1}, with some constant $C_j = C_j(s, \delta_1) > 0$, as follows

$$(3.37)\qquad (s + j\delta_1)D_t\underline{\alpha} q^j = (s + j\delta_1)\frac{2t - T}{t(T-t)}\underline{\alpha} e^{\delta_1 \underline{\alpha}} q^{j-1} \leq C_j q^{j-1}.$$

By maximal parabolic regularity (see [11]) we have

$$(3.38)\qquad \|q^j\|_{W_{\sigma_{j-1}}^{2,1}} \leq \tilde{C}_j \|q^{j-1}\|_{L^{\sigma_{j-1}}}$$

and using Sobolev type embedding from Lemma 2, we have that there exists a constant K_j such that

$$\|q^{j-1}\|_{L^{\sigma_{j-1}}} \leq K_j \|q^{j-1}\|_{W^{2,1}_{\sigma_{j-2}}}. \tag{3.39}$$

The sequence $(\sigma_m)_m$ is increasing to $+\infty$ and choose rank m_0 such that $\sigma_{m_0} > \frac{N+2}{2} \geq \sigma_{m_0-1}$. This implies that

$$W^{2,1}_{\sigma_{m_0}}(Q) \subset L^\infty(Q). \tag{3.40}$$

From (3.38), (3.39) and (3.40), and with the use of (3.17), we have that there exists a constant $C > 0$ such that

$$\begin{aligned} \|pe^{(s+m_0\delta_1)\underline{\alpha}}\|_{L^\infty(Q)} &= \|q^{m_0}\|_{L^\infty(Q)} \leq C\|q^0\|_{L^{\sigma_0}(Q)} = C\|pe^{s\underline{\alpha}}\|_{L^2(Q)} \\ &\leq C\|p_0 e^{s\overline{\alpha}}\|_{L^2(Q_{\omega_0})}. \end{aligned} \tag{3.41}$$

Remark 2. *In order to obtain the observability inequality we proceed in the classical manner, by multiplying scalarly in $L^2(\Omega)$ each equation of the system (3.1) by p_i and making use of dissipativity to find, for some constant $c > 0$ depending only on the coefficients of the system, the inequality:*

$$\frac{1}{2}\frac{d}{dt}\|p\|^2_{L^2(\Omega)} + c\|p\|^2_{L^2(\Omega)} \geq 0,$$

which gives

$$\|p(0)\|^2_{L^2(\Omega)} \leq \|p(t)\|^2_{L^2(\Omega)} e^{Ct}, t \in (0,T).$$

Consequently, for fixed $s > s_0$, we have that

$$\|p(0)\|^2_{L^2(\Omega)} \leq \frac{T}{2}\int_{\frac{T}{4}}^{\frac{3T}{4}} \|p(t)\|^2_{L^2(\Omega)} e^{Ct} dt \leq K(T,s)\int_0^T \|p(t)\|^2_{L^2(\Omega)} e^{2s\underline{\alpha}} dt.$$

Now, by Carleman estimate (3.34) we obtain the observability inequality, with a constant $C = C(T, s, \delta, M, \{\underline{\omega}_j\}_j)$:

$$\|p(0,\cdot)\|^2_{L^2(\Omega)} \leq C\int_{Q_{\omega_0}} |p_0|^2 e^{2s\overline{\alpha}} dx dt. \tag{3.42}$$

4. Linear system: null controllability

The main controllability result concerning linear system (2.6) is the following

Theorem 2. *Consider system (2.6) with coefficients in $\mathcal{E}_{M,\delta,\{\underline{\omega}_i\}_i}$. Then there exists a constant $C = C(M, \delta, \{\underline{\omega}_i\}_i)$ such that for all $z^0 \in H$ there exists $u^* \in L^2(0,T; L^2(\omega_0)) \cap L^\infty(Q_{\omega_0})$ which drives the corresponding solution to (2.6), $z = z^{u^*}$ in 0 i.e. satisfies $z(T,\cdot) = 0$ and satisfies the norm estimate*

$$\|u^* e^{-s\overline{\alpha}}\|_{L^2(0,T;L^2(\omega_0))} + \|u^*\|_{L^\infty(Q_{\omega_0})} \leq C\|z^0\|_{L^2(\Omega)}. \tag{4.1}$$

Proof.

$L^2(Q)$ *control.* In order to obtain norm estimates for the controls driving the trajectory to the linear system in 0, we consider a family of optimal control problems depending on a small parameter $\varepsilon > 0$:

$$\inf_{u\in L^2(Q_{\omega_0})} \frac{1}{2}\int_{Q_{\omega_0}} |u|^2 e^{-2s\overline{\alpha}} dxdt + \frac{1}{2\varepsilon}\int_{\Omega} |z(T,\cdot)|^2 dxdt, \tag{4.2}$$

with $z = z^u$ the solution of the linear controlled system (2.10). Classical results concerning optimal control with quadratic cost for parabolic equations insure existence of optimal control u^ε which by Pontriaghin maximum principle satisfy

$$u^\varepsilon = e^{2s\overline{\alpha}}\mathbf{B}^* p^\varepsilon = e^{2s\overline{\alpha}} p_0^\varepsilon|_{\omega_0}. \tag{4.3}$$

where p^ε is solution to the adjoint system:

$$\begin{cases} D_t p^\varepsilon = -\mathbf{A}p^\varepsilon - \mathbf{C}(t)p^\varepsilon - \mathbf{A}_0^*(t)p^\varepsilon, \\ p^\varepsilon(T) = -\frac{1}{\varepsilon} z^\varepsilon(T). \end{cases} \tag{4.4}$$

By cross multiplying the equations for $z^\varepsilon = z^{u^\varepsilon}$ and p^ε by p^ε respectively z^ε and integrating on Q we obtain:

$$\frac{d}{dt}\langle z^\varepsilon, p^\varepsilon\rangle_{L^2(\Omega)} = \langle (A+A_0+C)z^\varepsilon + Bu^\varepsilon, p^\varepsilon\rangle_{L^2(\Omega)} - \langle (A+A_0+C)^* p^\varepsilon, z^\varepsilon\rangle_{L^2(\Omega)}.$$

We integrate on $[0,T]$ and use the observability inequality (3.42)

$$\begin{aligned} &\frac{1}{\varepsilon}\|z^\varepsilon(T,\cdot)\|^2_{L^2(\Omega)} + \langle u^\varepsilon, B^* p^\varepsilon\rangle_{L^2(Q)} = -\langle z^\varepsilon(0,\cdot), p^\varepsilon(0,\cdot)\rangle_{L^2(\Omega)} \\ &\le \|z^0\|_{L^2(\Omega)}\|p(0,\cdot)\|_{L^2(\Omega)} \le C\|z^0\|_{L^2(\Omega)}\left(\int_{Q_{\omega_0}} |p_0^\varepsilon|^2 e^{2s\overline{\alpha}} dxdt\right)^{\frac{1}{2}}. \end{aligned}$$

Since $\langle u^\varepsilon, B^* p^\varepsilon\rangle_{L^2(Q)} = \int_{Q_{\omega_0}} |p_0^\varepsilon|^2 e^{2s\overline{\alpha}} dxdt$, using appropriately balanced Young's inequality, we find that

$$\frac{1}{\varepsilon}\|z^\varepsilon(T,\cdot)\|^2_{L^2(\Omega)} + \frac{1}{2}\int_{Q_{\omega_0}} |p_0^\varepsilon|^2 e^{2s\overline{\alpha}} dxdt \le C\|z^0\|^2_{L^2(\Omega)}, \tag{4.5}$$

gives by (4.3) the following estimate the sequence of optimal controls $(u^\varepsilon)_\varepsilon$ and final state:

$$\frac{1}{\varepsilon}\|z^\varepsilon(T,\cdot)\|^2_{L^2(\Omega)} + \frac{1}{2}\int_{Q_{\omega_0}} |u^\varepsilon|^2 e^{-2s\overline{\alpha}} dxdt \le C\|z^0\|^2_{L^2(\Omega)}. \tag{4.6}$$

Now, this L^2 bound for the sequence $(u^\varepsilon)_\varepsilon$, allows to extract a subsequence, denoted for simplicity also $(u^\varepsilon)_\varepsilon$ weakly convergent in $L^2(Q)$ to a limit u^*.

Write the corresponding solutions $(z^\varepsilon)_\varepsilon$ as

$$z^\varepsilon = w^\varepsilon + v$$

where w^ε is solution to (2.6) with initial data $w^\varepsilon(0) = 0$ and v solution to homogeneous equation

$$D_t v = \mathbf{A}v + (A_0 + C)v = 0,\ v(0) = z^\varepsilon(0) = z^0.$$

We have that the sequence $(w^\varepsilon)_\varepsilon$ is bounded in $L^2(0,T;D(\mathbf{A}))$ and the sequence of derivatives $(D_t w^\varepsilon)_\varepsilon$ is bounded in $L^2(0,T;L^2(\Omega))$. By Aubin's theorem we can extract a subsequence, denoted also $(w^\varepsilon)_\varepsilon$, strongly convergent in $L^2(0,T;H_0^1(\Omega))$ to $w \in L^2(0,T;H_0^1(\Omega)) \cap L^2(0,T;D(\mathbf{A}))$. Consequently (z^ε) is strongly convergent in $L^2(0,T;H_0^1(\Omega))$ to $z \in L^2(0,T;H_0^1(\Omega))$. We may now pass to the limit in the weak formulation of solutions to (2.6), (2.10); thus, for some test function $\boldsymbol{\varphi} \in [H_0^1(\Omega)]^{n+1}$, we have

$$(4.7) \quad \begin{cases} \langle z^\varepsilon(t,\cdot), \boldsymbol{\varphi}\rangle_{L^2(\Omega)} - \langle z^\varepsilon(0,\cdot), \boldsymbol{\varphi}\rangle_{L^2(\Omega)} + \displaystyle\int_0^t \langle \nabla z^\varepsilon(\tau,\cdot), \nabla\boldsymbol{\varphi}\rangle_{L^2(\Omega)} d\tau \\ \\ + \displaystyle\int_0^t \langle (A_0 + C) z^\varepsilon, \boldsymbol{\varphi}\rangle_{L^2(\Omega)} d\tau = \displaystyle\int_{(0,t)\times\omega_0} u^\varepsilon \boldsymbol{\varphi} dx d\tau, \\ z^\varepsilon(0,\cdot) := z^0, \end{cases}$$

and we find that $z \in L^2(Q)$ solution for the problem (2.10) with initial datum $z^0 \in L^2(\Omega)$. In fact, by Arzelà-Ascoli theorem $w^\varepsilon \to w$ in $C([0,T], L^2(\Omega))$ and thus $z(T) = 0$ and by weak lower semicontinuity of the L^2 norm we also have the following estimate for the control driving the solution to 0:

$$(4.8) \qquad \int_{Q_{\omega_0}} |u^*|^2 e^{-2s\overline{\alpha}} dx dt \leq C \|z^0\|^2_{L^2(\Omega)}.$$

where $C = C(T, s, s_1, M, \delta, \{\underline{\omega}_j\}_j)$.

$L^\infty(Q)$- *control.* Regarding the L^∞ norm estimates for the sequence $(u^\varepsilon)_\varepsilon$ and also for u^* we will use the results from the previous section §3:

$$(4.9) \quad \|u^\varepsilon e^{-2s\overline{\alpha}+(s+m_0\delta_1)\underline{\alpha}}\|_{L^\infty(Q_{\omega_0})} = \|p_0^\varepsilon e^{(s+m_0\delta_1)\underline{\alpha}}\|_{L^\infty(Q_{\omega_0})} \leq C\|z^0\|_{L^2(Q)}.$$

Now we see that we could start from the beginning with λ big enough such that (3.16) holds and in consequence

$$2s\overline{\alpha} \leq (s + m_0\delta_1)\underline{\alpha}.$$

As $-2s\overline{\alpha} + (s+m_0\delta_1)\underline{\alpha} > 0$, by passing to L^∞ weak-* limit in (4.9), we find that

$$(4.10) \qquad \|u^*\|_{L^\infty(Q_{\omega_0})} \leq \|u^* e^{-2s\overline{\alpha}+(s+m_0\delta_1)\underline{\alpha}}\|_{L^\infty(Q_{\omega_0})} \leq C\|z^0\|_{L^2(Q)},$$

which concludes (4.1).

■

5. Nonlinear system: local exact controllability

We prove in this section the following local controllability result concerning system (2.1):

Theorem 3. *Suppose $\overline{y}$ is a stationary state, i.e. solution to (2.2), and that the functions $f_j, j \in \overline{0,n}$ satisfy hypotheses (H1), (H2). Then, for all $\beta_0 > 0$ there exist $\zeta_0 = \zeta_0(\beta_0) > 0$ and $C = C(\beta_0, \{\underline{\omega}_i\}_i, \overline{y})$ such that if $\|y^u(0) - \overline{y}\| < \zeta_0$ there exists a control $u \in L^\infty(Q)$ satisfying*

$$\|u\|_{L^\infty(Q)} \leq C\|y^u(0) - \overline{y}\|_{L^\infty(\Omega)}$$

and

$$y^u(T,\cdot) = \overline{y},$$

with

$$\|y(t,\cdot) - \overline{y}\|_{L^\infty} \leq \beta_0,\ t \in [0,T].$$

Proof. The approach to the local null controllability of the system around the stationary state is based on the Kakutani fixed point theorem.

For this aim, with a solution y to (2.1), we consider the system satisfied by $z := y - \overline{y}$, written as a linear system

$$(5.1)\quad \begin{cases} D_t z_0 - \Delta z_0 = c_0^z(t,x) z_0 + \chi_{\omega_0} u, & (0,T)\times\Omega, \\ D_t z_i - \Delta z_i = a_{i0}^z(t,x) z_0 + c_i^z(t,x) z_i,\ i \in \overline{1,n}, & (0,T)\times\Omega, \\ z_0 = \ldots = z_n = 0, & (0,T)\times\partial\Omega, \\ z(0,x) = z^0(x) := y(0,x) - \overline{y}(x) & x\in\Omega, \end{cases}$$

where the nonlinearity is hidden into the coupling coefficients which are defined by:

$$a_{i0}^z(t,x) := \int_0^1 \frac{\partial}{\partial y_0} f_i(x, \overline{y}_0(x) + \tau z_0(t,x), \overline{y}_i(x) + \tau z_i(t,x)) d\tau,\ i \in \overline{1,n}$$
$$c_j^z(t,x) := \int_0^1 \frac{\partial}{\partial y_j} f_j(x, \overline{y}_0(x) + \tau z_0(t,x), \overline{y}_j(x) + \tau z_j(t,x)) d\tau,\ j \in \overline{0,n}. \quad (5.2)$$

Observe that $\{a_{i0}^0, c_j^0\}_{i\in\overline{1,n}, j\in\overline{0,n}}$ are the coefficients of the linearized system around the stationary solution $\overline{y}$ as

$$a_{i0}^0(x) = \frac{\partial}{\partial y_0} f_i(x, \overline{y}_0(x), \overline{y}_i(x)),$$

$$c_i^0 = \frac{\partial}{\partial y_i} f_i(x, \overline{y}_0(x), \overline{y}_i(x)), c_0^0 = \frac{\partial}{\partial y_0} f_0(x, \overline{y}_0(x)).$$

We see now that hypotheses (2.4) and (2.5) tell us that we may choose $M_0, \delta_0 > 0$ and $\underline{\omega}_i \subset\subset (\omega_i \cap \omega_0) \setminus \bigcup_{j\neq 0,i} \omega_j$ such that

$$(5.3)\quad \{a_{i0}^0, c_j^0\}_{i\in\overline{1,n}, j\in\overline{0,n}} \in \mathcal{E}_{M_0, \delta_0, \{\underline{\omega}_i\}_i}.$$

Let $\beta > 0$ and define $\mathcal{M}_\beta$ to be:

$$(5.4)\quad \mathcal{M}_\beta = \{\tilde{z} \in L^\infty(Q) : \|\tilde{z}\|_{L^\infty(Q)} \le \beta\}.$$

For $\tilde{z} \in \mathcal{M}_\beta$, we consider the coefficients $a_{i0}^{\tilde{z}}(x)$, $c_j^{\tilde{z}}(x)$ defined as in (5.2) with z replaced by $\tilde{z}$.

Observe now that we may choose $\beta_0 > 0$ small enough such that if $\tilde{z} \in \mathcal{M}_{\beta_0}$ we have

$$(5.5)\quad \{a_{i0}^{\tilde{z}}, c_j^{\tilde{z}}\}_{i\in\overline{1,n}, j\in\overline{0,n}} \in \mathcal{E}_{2M_0, \frac{\delta_0}{2}, \{\underline{\omega}_i\}_i}.$$

Consider now the linear system (5.1) with coefficients $\{a_{i0}^{\tilde{z}}, c_j^{\tilde{z}}\}$:

$$(5.6)\quad \begin{cases} D_t z_0 - \Delta z_0 = c_0^{\tilde{z}}(t,x) z_0 + \chi_{\omega_0} u, & (0,T)\times\Omega, \\ D_t z_i - \Delta z_i = a_{i0}^{\tilde{z}}(t,x) z_0 + c_i^{\tilde{z}}(t,x) z_i,\ i \in \overline{1,n}, & (0,T)\times\Omega, \\ z_0 = \ldots = z_n = 0, & (0,T)\times\partial\Omega, \\ z(0,x) = z^0(x) & x\in\Omega. \end{cases}$$

The linear problem (5.6) may be reformulated as:

$$(5.7)\quad \begin{cases} D_t z = \mathbf{A} z + \mathbf{A}_0^{\tilde{\mathbf{z}}}(t) z + \mathbf{C}^{\tilde{\mathbf{z}}}(t) z + \mathbf{B} u, \quad t > 0, \\ z(0) = z^0, \end{cases}$$

where $\mathbf{C}^{\tilde{\mathbf{z}}}(t)z = C_0^{\tilde{z}}(t,\cdot)z(\cdot)$ and $\mathbf{A}_{\mathbf{0}}^{\tilde{\mathbf{z}}}(t)z = A_0^{\tilde{z}}(t,\cdot)z(\cdot)$ where $C_0^{\tilde{z}}(t,x) = diag(c_i^{\tilde{z}}(t,x))_{i=\overline{0,n}}$ and the coupling matrix

$$A_0^{\tilde{z}}(t,x) = (0, a_{10}^{\tilde{z}}(t,x), \dots, a_{n0}^{\tilde{z}}(t,x))^\top \cdot (1,0,\dots,0).$$

Theorem 2 says that for $\tilde{z} \in \mathcal{M}_{\beta_0}$ there exists a control $u^* = u^*(\tilde{z}) \in L^2(0,T;L^2(\omega_0)) \cap L^\infty(Q_{\omega_0})$ satisfying the norm estimate

$$\begin{aligned} J(u^*) &:= \|u^* e^{-s\overline{\alpha}}\|_{L^2(0,T;L^2(\omega_0))} + \|u^*\|_{L^\infty(Q_{\omega_0})} \\ &\leq C(2M_0, \delta_0/2, \{\underline{\omega}_i\}_i)\|z^0\|_{L^2(\Omega)}, \end{aligned} \tag{5.8}$$

and driving the solution $z^{u^*,\tilde{z}}$ of the linear system (5.6) in zero : $z^{u^*,\tilde{z}}(T) = 0$. Observe that J is a norm in the space $\mathcal{U}^* := L^2_{e^{-s\overline{\alpha}}} \cap L^\infty(Q_{\omega_0})$.

We will write

$$z^{u,\tilde{z}} = T_1^{\tilde{z}}(z^0) + T_2^{\tilde{z}}(u), \tag{5.9}$$

where the first term is the solution to problem (5.6) with initial data z^0 and the second term is the solution to system (5.6) with initial datum zero and control u. Let us denote by

$$S_1(z^0) = e^{t\mathbf{A}}z^0,\ S_2h = e^{t\mathbf{A}} * h = \int_0^t e^{(t-s)\mathbf{A}}h(s)ds, \tag{5.10}$$

where $h \in L^2(0,t;[L^2(\Omega)]^{n+1})$. With these notations

$$z^{u,\tilde{z}} = T_1^{\tilde{z}}(z^0) + T_2^{\tilde{z}}(u) = S_1(z^0) + S_2(A_0^{\tilde{z}}z^{u,\tilde{z}} + C_0^{\tilde{z}}z^{u,\tilde{z}} + Bu). \tag{5.11}$$

Fix an initial datum $z^0 \in L^\infty(\Omega)$. We define now the following set-valued map, associated to z^0:

$$\begin{aligned} &F_{z^0} : \mathcal{M}_{\beta_0} \to 2^{L^\infty(Q)} \\ &F_{z^0}(\tilde{z}) = \{z^{u,\tilde{z}} : u \text{ satisfies (5.8) and } z^{u,\tilde{z}}(T) = 0\} \\ &\qquad = \{T_1^{\tilde{z}}(z^0) + T_2^{\tilde{z}}(u) :\ z^{u,\tilde{z}}(T) = 0, J(u) \leq K\|z^0\|_{L^2}\}, \end{aligned} \tag{5.12}$$

where by K we denoted the constant in (5.8), $K = C(2M_0, \delta_0/2, \{\underline{\omega}_i\}_i)$.

In order to obtain local controllability of the nonlinear system it is enough to find a fixed point for F_{z^0}. We achieve this goal by applying Kakutani fixed point theorem to F_{z^0} in $\mathcal{M}_{\beta_0}$; we have thus to verify the following statements:

i) For every $\widetilde{z} \in \mathcal{M}$, $F_{z^0}(\widetilde{z})$ is a nonempty, closed and convex subset of $L^\infty(Q)$;

Observe that $z^{u^*(\widetilde{z})} \in F_{z^0}(\widetilde{z})$ and thus $F_{z^0}(\widetilde{z}) \neq \emptyset$. Convexity comes from linearity of T_2 and convexity of J.

To prove that $F_{z^0}(\widetilde{z})$ is closed, suppose $z^m \in F_{z^0}(\widetilde{z})$, $z^m \to z$ in L^∞. We have to prove that $z \in F_{z^0}(\widetilde{z})$. Indeed, we have that

$$z^m = T_1^{\tilde{z}}(z^0) + T_2^{\tilde{z}}(u^m)$$

for some controls $u^m \in \mathcal{U}^*$ satisfying estimate $J(u^m) \leq K\|z^0\|_{L^2}$. We may now invoke Aubin-Lions and Ascoli-Arzelà compactness results (see *e.g.* [15]) applied to the solution operator of a parabolic initial boundary value problem and thus to say that T_2 is a compact operator from $L^2(0,T;L^2(\Omega_{\omega_0}))$ to $C([0,T];[L^2(\Omega)]^{n+1}) \cap L^2(0,T;[H^1_0(\Omega)]^{n+1})$. Thus, extracting subsequence $u^m \rightharpoonup u$ weakly in $L^2(Q_{\omega_0})$ we find

$$z^m \to z \text{ in } C([0,T];[L^2(\Omega)]^{n+1}) \cap L^2(0,T;[H^1_0(\Omega)]^{n+1})$$

with $z(T)=0$ since $z_m(T)=0$. Thus $z \in F_{z^0}(\widetilde{z})$.

ii) There exists $\zeta_0 = \zeta_0(\beta_0)$ such that for $\|z^0\|_{L^\infty(\Omega)} < \zeta_0$ we have

$$F_{z^0}(\mathcal{M}_{\beta_0}) \subset \mathcal{M}_{\beta_0}.$$

This follows from the a priori estimates for solutions to initial boundary value problems for parabolic systems:

$$\|T^{\tilde{z}}_1(z^0)\|_{L^\infty(\Omega)} \le C_1(\|\tilde{z}\|_{L^\infty})\|z^0\|_{L^\infty(\Omega)},$$

$$\|T^{\tilde{z}}_2(u)\|_{L^\infty(\Omega)} \le C_2(\|\tilde{z}\|_{L^\infty})\|u\|_{L^\infty(Q_{\omega_0})}$$

and from the remark that both constants depend in fact uniformly on the L^∞ norm of the couplking coefficients and thus depend uniformly on the norm of $\tilde{z}$ in L^∞.

iii) The set $F_{z^0}(\mathcal{M}_{\beta_0})$ is imbedded into a convex and compact subset of $\mathcal{M}_{\beta_0}$.

Indeed, as $\mathcal{M}_{\beta_0}$ is closed and convex, it is enough to prove that $F_{z^0}(\mathcal{M}_{\beta_0})$ is relatively compact in L^∞ topology. For this, take a sequence $z^m \in F_{z^0}(\mathcal{M}_{\beta_0})$. Correspondingly, there exist $\tilde{z}_m \in \mathcal{M}_{\beta_0}$ with $z^m \in F_{z^0}(\tilde{z}^m)$. Take corresponding controls $u^m \in \mathcal{U}^*$ such that (see definition of F_{z^0} and (5.11)):

$$z^m = T^{\tilde{z}^m}_1(z^0) + T^{\tilde{z}^m}_2(u^m) = S_1(z^0) + S_2(A^{\tilde{z}^m}_0 z^m + C^{\tilde{z}^m}_0 z^m + Bu^m). \tag{5.13}$$

We have the following bounded sequences

- $\tilde{z}^m \in \mathcal{M}_{\beta_0}$ and so $A^{\tilde{z}^m}_0(Q), C^{\tilde{z}^m}_0(Q)$ are bounded in L^∞;
- $z^m \in \mathcal{M}_{\beta_0}$ and is thus bounded in $L^\infty(Q)$;
- $u^m \in \mathcal{U}^*$ is bounded in $L^\infty(Q)$.

Consequently $A^{\tilde{z}^m}_0 z^m + C^{\tilde{z}^m}_0 z^m + Bu^m$ is bounded in $L^p(Q), p > 1$. By parabolic regularity (see [11]), $S_2(A^{\tilde{z}^m}_0 z^m + C^{\tilde{z}^m}_0 z^m + Bu^m)$ is bounded in any $W^{2,1}_p, \forall 1 < p < \infty$ (the space of anisotropic Sobolev functions). For p big enough we have $W^{2,1}_p \subset C^{0,\alpha}(\overline{Q})$ for some $0 < \alpha < 1$ (the space of Hölder continuous functions). $C^{0,\alpha}(\overline{Q})$ is compactly imbedded in $C(\overline{Q})$. Consequently (z^m) is a relatively compact sequence in $L^\infty(Q)$.

iv) F_{z^0} is upper semi-continuous, *i.e.* if $z^m \to z$, $\tilde{z}^m \to \tilde{z}$ in L^∞ and $z^m \in F_{z^0}(\tilde{z}^m)$ then $z \in F_{z^0}(\tilde{z})$.

Indeed we have (see (5.2)) that $A^{\tilde{z}^m}_0 \to A^{\tilde{z}}_0, C^{\tilde{z}^m}_0 \to C^{\tilde{z}}_0$ in L^∞ and as z^m is relatively compact in $C([0,T];[L^2(\Omega)]^{n+1})$ we may pas to the limit in (5.13) and find that $z \in F_{z^0}(\tilde{z})$.

Now we conclude the proof by Kakutani fixed point theorem, which insures existence of $z \in \mathcal{M}_{\beta_0}$ such that $z \in F_{z^0}(z)$ *i.e.* there exists $\overline{u} \in \mathcal{U}^*$ such that $z^{\overline{u},z} = z$. In conclusion $y^{\overline{u}} := \overline{y} + z$ is the solution to the controlled system (2.1) with control $\overline{u}$ satisfying $y^{\overline{u}}(T) = \overline{y}$. ∎

6. Parabolic systems with tree-like couplings. Null controllability.

We describe in the following what we mean by a system with tree-like couplings. This would be a parabolic system of the form

(6.1)
$$\begin{cases} D_t z_0 - \Delta z_0 = c_0(t,x)z_0 + \chi_{\omega_0} u, & \text{in } (0,T)\times\Omega, \\ D_t z_i - \Delta z_i = a_{i\mathbf{k}(i)}(t,x)z_{\mathbf{k}(i)} + c_i(t,x)z_i,\ i\in\overline{1,n}, & \text{in } (0,T)\times\Omega, \\ z_0 = \dots = z_n = 0, & \text{on } (0,T)\times\partial\Omega, \\ z(0,\cdot) = z^0, \end{cases}$$

with the following assumptions on the function $\mathbf{k} : \{1,\dots,n\} \to \{0,1,\dots,n\}$:

(6.2)
$$\forall i \in \{1,\dots,n\}, \exists m = m(i), 1 \le m \le n-1, (\mathbf{k}\circ)^m(i) = \mathbf{k}\circ\dots\circ\mathbf{k}(i) = 0.$$

The linear problem (6.1) may be reformulated as:

$$\begin{cases} D_t z = \mathbf{A}z + \mathbf{A_0}(t)z + \mathbf{C}(t)z + \mathbf{B}u, \quad t>0, \\ z(0) = z^0, \end{cases} \tag{6.3}$$

where $\mathbf{C}(t)z = C_0(t,\cdot)z(\cdot)$ and $\mathbf{A_0}(t)z = A_0(t,\cdot)z(\cdot)$ where $C_0(t,x) = diag(c_i(t,x))_{i=\overline{0,n}}$ and the coupling matrix

$$A_0(t,x) = (a_{il})_{i,l\in\overline{1,n}} = (a_{i\mathbf{k}(i)}\delta_{l\mathbf{k}(i)})_{i,l\in\overline{1,n}},$$

where we denoted by δ_{lj} the Kronecker symbol. Denote by

$$\mathbf{I}_j = \mathbf{k}^{-1}(j) = \{i\in\overline{1,n} : \mathbf{k}(i) = j\}.$$

Fix now a family of open subsets $\omega_i \subset \Omega, i\in\overline{1,n}$ such that

$$D_i := \omega_i \cap \omega_{\mathbf{k}(i)} \cap \dots \cap \omega_{(\mathbf{k}\circ)^{m(i)}} \ne \emptyset. \tag{6.4}$$

$$D_i \setminus \bigcup_{j\ne i, \mathbf{k}(j)=\mathbf{k(i)}} \omega_j \ne \emptyset. \tag{6.5}$$

Choose further a family of open subsets $\{\underline{\omega}_j\}_{j\in\overline{0,n}}$ with the properties

$$\underline{\omega}_0 \subset\subset \omega_0, \quad \underline{\omega}_i \subset\subset D_i \setminus \bigcup_{l\ne i,\mathbf{k}(l)=\mathbf{k(i)}} \omega_l, \tag{6.6}$$

$$\underline{\omega}_i \subset\subset \underline{\omega}_{\mathbf{k}(i)} \subset\subset \underline{\omega}_0,\ i\in\overline{1,n}. \tag{6.7}$$

For $M,\delta>0$, and the family of open subsets described above $\{\underline{\omega}_i\}_i$, we introduce the following classes of coefficients sets:

(6.8)
$$\mathcal{E}_{M,\delta,\{\underline{\omega}_i\}_i,\mathbf{k}} = \Big\{ E = \{a_{i\mathbf{k}(i)}, c_j\}_{i\in\overline{1,n},j\in\overline{0,n}} : a_{i\mathbf{k}(i)}, c_j \in L^\infty(Q),$$
$$\|a_{i\mathbf{k}(i)}\|_{L^\infty}, \|c_j\|_{L^\infty} \le M, a_{i\mathbf{k}(i)} = 0 \text{ in } Q\setminus Q_{\omega_i}, \text{ and } |a_{i\mathbf{k}(i)}| \ge \delta \text{ on } Q_{\underline{\omega}_i} \Big\}.$$

In order to study controllability we consider the system adjoint to system (6.1):

(6.9)
$$\begin{cases} -D_t p_j - \Delta p_j - c_j(t,x)p_j = \sum_{l,\mathbf{k}(l)=j} a_{lj}(t,x)p_l = \mathcal{N}_j(t,x),\ j\in\overline{0,n}, \text{ in } Q,\\ p_0 = \dots = p_n = 0, \text{ on } (0,T)\times\partial\Omega, \end{cases}$$

where for simplicity of further calculations we denoted by

$$\mathcal{N}_j(t,x) = \sum_{l,\mathbf{k}(l)=j} a_{lj}(t,x)p_l(t,x).$$

As we have seen in the previous sections all controllability results have as essential ingredient an appropriate Carleman inequality for the adjoint system. For obtaing such estimates it is essential to have corresponding auxiliary functions which appear in the construction of the weights. We describe this in what follows

Consider again open subsets

$$\tilde{\omega}_j \subset\subset \underline{\omega}_j, j\in\overline{0,n},$$

and auxiliary functions

$$\eta_j \in C^2(\overline{\Omega}),\ 0<\eta_j \text{ in } \Omega,\ \eta_j|_{\partial\Omega}=0, \{x\in\overline{\Omega} : |\nabla\eta_j(x)|=0\}\subset\subset\tilde{\omega}_j, j\in\overline{0,n}.$$

We construct now the weight functions entering the various Carleman estimates, with the following properties:

i) $\psi_{j,f}, j\in\overline{0,n}, \mathbf{I}_j\neq\emptyset$, $\psi_{i,s}, i\in\overline{1,n}$ are defined by

$$\psi_{j,f} := \eta_j + K_j, \quad \psi_{i,s} := \eta_i + \tilde{K}_i \tag{6.10}$$

for some fixed positive constants $K_j, \tilde{K}_i > 0$ and such that for a fixed $\epsilon>0$ we have

$$\psi_{i,s} > \sup_{\overline{\Omega}} \psi_{j,f} + 2\epsilon, \forall i\in\mathbf{I}_j,\ \mathbf{I}_j\neq\emptyset; \tag{6.11}$$

$$\psi_{i,f} > \sup\{\psi_{l,s} : \mathbf{k}(l)=\mathbf{k}(i)\} + 2\epsilon, \forall i\in\overline{1,n}, \mathbf{I}_i\neq\emptyset; \tag{6.12}$$

ii)

$$\frac{\sup\psi_{j,f}}{\inf\psi_{j,f}} < \frac{8}{7}, \frac{\sup\psi_{i,s}}{\inf\psi_{i,s}} < \frac{8}{7}; \tag{6.13}$$

iii) For $j\in\overline{0,n}$ such that $\mathbf{I}_j\neq\emptyset$ we define

$$\overline{\psi}_j = \sup\{\psi_{j,f}(x), \psi_{i,s}(x) : i\in\mathbf{I}_j, x\in\Omega\} + \epsilon, \tag{6.14}$$

$$\underline{\psi}_j = \inf\{\psi_{j,f}(x), \psi_{i,s}(x) : i\in\mathbf{I}_j, x\in\Omega\} - \epsilon. \tag{6.15}$$

iv) Denote by $\overline{\psi} = \sup\{\overline{\psi}_j : \mathbf{I}_j\neq\emptyset\}$ and $\underline{\psi} = \inf\{\underline{\psi}_j : \mathbf{I}_j\neq\emptyset\}$ and

$$\overline{\varphi}_j(t) = \overline{\varphi}_j^\lambda(t) := \frac{e^{\lambda\overline{\psi}_j}}{t(T-t)}, \quad \overline{\alpha}_j(t) = \overline{\alpha}_j^\lambda(t) := \frac{e^{\lambda\overline{\psi}_j} - e^{1.5\lambda\overline{\psi}}}{t(T-t)}, \tag{6.16}$$

$$\underline{\varphi}_j(t) = \underline{\varphi}_j^\lambda(t) := \frac{e^{\lambda\underline{\psi}_j}}{t(T-t)}, \quad \underline{\alpha}_j(t) = \underline{\alpha}_j^\lambda(t) := \frac{e^{\lambda\underline{\psi}_j} - e^{1.5\lambda\overline{\psi}}}{t(T-t)}. \tag{6.17}$$

$$\underline{\alpha}(t) = \frac{e^{\lambda\underline{\psi}} - e^{1.5\lambda\overline{\psi}}}{t(T-t)},\ \overline{\alpha}(t) = \frac{e^{\lambda\overline{\psi}} - e^{1.5\lambda\overline{\psi}}}{t(T-t)} \tag{6.18}$$

Remark 3. *Observe that this construction of the weight functions allows saying that*

$$\overline{\psi}_j < \underline{\psi}_i, i \in \boldsymbol{I}_j, \boldsymbol{I}_j \neq \emptyset,$$

and thus, given $\theta > 0$ there exists $s(\theta)$ such that for $s > s(\theta)$ we have

$$e^{s\overline{\alpha}_j(t)} \leq \theta e^{s\underline{\alpha}_i(t)}, i \in \boldsymbol{I}_j, \boldsymbol{I}_j \neq \emptyset, t \in [0,T]. \tag{6.19}$$

The Carleman estimates we establish now in the tree coupling case are given in the following theorem:

Theorem 4. *Suppose that the coupling coefficients in (6.9) satisfy*

$$\{a_{i\boldsymbol{k}(i)}, c_j\}_{i\in\overline{1,n}, j\in\overline{0,n}} \in \mathcal{E}_{M,\delta,\{\underline{\omega}_i\}_i,\boldsymbol{k}}.$$

Then there exist constants λ_0, s_0 such that for $\lambda > \lambda_0$ there exists a constant $C > 0$ depending on $(M, \delta, \{\underline{\omega}_i\}_i, \lambda)$ such that, for any $s \geq s_0$, the following inequality holds:

$$\begin{aligned} &\int_Q (|D_t p|^2 + |D^2 p|^2 + |Dp|^2 + |p|^2) e^{2s\underline{\alpha}} dxdt \\ &\leq C \int_{Q_{\omega_0}} |p_0|^2 e^{2s\overline{\alpha}} dxdt \end{aligned} \tag{6.20}$$

for all $p \in H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega))$ solution of (6.9).

Moreover, there exists $m_0 \in \mathbb{N}$ and $\delta_1 > 0$ such that we have the following $L^\infty - L^2$ Carleman estimate

$$\|p e^{(s+m_0\delta_1)\underline{\alpha}}\|_{L^\infty(Q)} \leq C \|p_0 e^{s\overline{\alpha}}\|_{L^2(Q_{\omega_0})}. \tag{6.21}$$

Proof. For $j \in \overline{0,n}$ we write separately Carleman inequalities for the case $\mathbf{I}_j \neq \emptyset$ and respectively for the case $\mathbf{I}_j = \emptyset$. If $j \in \overline{0,n}$ is such that $\mathbf{I}_j \neq \emptyset$ we treat the equations satisfied by p_j and $p_l, l \in \mathbf{I}_j$ as a nonhomogeneous adjoin system, as in the star-like couplings (3.4), while in the case $\mathbf{I}_j = \emptyset$ we have to deal with homogeneous parabolic equations:

$$\begin{cases} -D_t p_j - \Delta p_j - c_j(t,x)p_j = \sum_{l,\mathbf{k}(l)=j} a_{lj}(t,x)p_l, & \text{in } (0,T)\times\Omega, \\ -D_t p_l - \Delta p_l - c_l(t,x)p_l = \mathcal{N}_l(t,x), & l \in \mathbf{I}_j. \end{cases} \tag{6.22}$$

For the case $\mathbf{I}_j \neq \emptyset$ a Carleman estimate, which is an immediate consequence to intermediate estimate (3.21), states that there exists $\overline{s}_j$ and $C > 0$ not depending on s such that for $s > \overline{s}_j$ we have

$$
\begin{aligned}
&\int_Q (|D_t p_j|^2 + |D^2 p_j|^2 + |Dp_j|^2 + |p_j|^2) e^{2s\underline{\alpha}_j} dxdt \\
&+ \int_Q \left[\sum_{i\in \mathbf{I}_j} (|D_t p_i|^2 + |D^2 p_i|^2 + |Dp_i|^2 + |p_i|^2) \right] e^{2s\underline{\alpha}_j} dxdt \\
&\le C \left[\int_{Q_{\underline{\omega}_j}} |p_j|^2 e^{2s\overline{\alpha}_j} dxdt + \sum_{i\in \mathbf{I}_j} \int_{Q_{\underline{\omega}_i}} |p_i|^2 e^{2s\overline{\alpha}_j} \right] \\
&+ C \sum_{i\in \mathbf{I}_j} \int_Q |\mathcal{N}_i(t,x)|^2 e^{2s\overline{\alpha}_j} dxdt \\
&\le C \left[\int_{Q_{\underline{\omega}_j}} |p_j|^2 e^{2s\overline{\alpha}_j} dxdt + \sum_{i\in \mathbf{I}_j} \int_{Q_{\underline{\omega}_i}} |p_i|^2 e^{2s\overline{\alpha}_i} \right] dxdt \\
&+ C \sum_{i\in \mathbf{I}_j, l\in \mathbf{I}_i} \int_Q \theta |p_l(t,x)|^2 e^{2s\underline{\alpha}_l} dxdt,
\end{aligned} \tag{6.23}
$$

where we have used Remark 3 in order to say that $e^{2s\overline{\alpha}_j} \le \theta e^{2s\underline{\alpha}_i} \le \theta e^{2s\underline{\alpha}_l}$ for $\theta > 0$ to be fixed later and $s > s(\theta)$ big enough.

In the case $\mathbf{I}_j = \emptyset$, we write the Carleman estimate for the homogeneous equation

$$
-D_t p_j - \Delta p_j - c_j(t,x) p_j = 0
$$

So, there exist constants $\overline{s}_j > 0$ and $C > 0$ such that for $s > \overline{s}_j$

$$
\begin{aligned}
&\int_Q (|D_t p_j|^2 + |D^2 p_j|^2 + |Dp_j|^2 + |p_j|^2) e^{2s\underline{\alpha}_j} dxdt \\
&\le C \int_{Q_{\underline{\omega}_j}} |p_j|^2 e^{2s\overline{\alpha}_j} dxdt.
\end{aligned} \tag{6.24}
$$

We add now estimates (6.23) and (6.24) and we obtain for some constant $C > 0$ and $s > \max_j \overline{s}_j$:

$$
\begin{aligned}
&\sum_{j\in\overline{0,n}} \int_Q (|D_t p_j|^2 + |D^2 p_j|^2 + |Dp_j|^2 + |p_j|^2) e^{2s\underline{\alpha}_j} dxdt \le \\
&C \left[\sum_{j\in\overline{0,n}} \int_{Q_{\underline{\omega}_j}} |p_j|^2 e^{2s\overline{\alpha}_j} dxdt + \sum_{j\in\overline{1,n}} \int_Q \theta |p_j(t,x)|^2 e^{2s\underline{\alpha}_j} \right] dxdt.
\end{aligned} \tag{6.25}
$$

Choosing θ small enough we that the integrals on Q in the right side may be absorbed in the left side of the inequality and obtain

$$
\begin{aligned}
&\sum_{j\in\overline{0,n}} \int_Q (|D_t p_j|^2 + |D^2 p_j|^2 + |Dp_j|^2 + |p_j|^2) e^{2s\underline{\alpha}_j} dxdt \\
&\le C \sum_{j\in\overline{0,n}} \int_{Q_{\underline{\omega}_j}} |p_j|^2 e^{2s\overline{\alpha}_j} dxdt.
\end{aligned} \tag{6.26}
$$

Observe now that for $j \geq 0$, by (6.2) there exists $m = m(j)$ and the sequence $j_0 = j, j_1 = \mathbf{k}(j_0), \ldots, j_m = (\mathbf{k}\circ)^m(j) = 0$. Now, by (6.4), (6.5), (6.6), (6.7), by looking only in the subdomains $\underline{\omega}_{j_l}, l \in \overline{0,m}$ we find a sequence of equations for $l \in \overline{0, m-1}$, forming cascade like system:

(6.27)
$$-D_t p_{j_{l+1}} - \Delta p_{j_{l+1}} - c_{j_{l+1}}(t,x) p_{j_{l+1}} = a_{j_l, j_{l+1}}(t,x) p_{j_l}, \text{ in } (0,T) \times \underline{\omega}_{j_{l+1}}.$$

Now, as $\underline{\omega}_{j_l} \subset\subset \underline{\omega}_{j_{l+1}}$ we find, as in the §3

$$\int_{Q_{\underline{\omega}_{j_l}}} |p_{j_l}|^2 e^{2s\overline{\alpha}_{j_l}} dxdt \leq C \int_{Q_{\underline{\omega}_{j_{l+1}}}} |p_{j_{l+1}}|^2 e^{2s\overline{\alpha}_{j_{l+1}}} dxdt. \tag{6.28}$$

Consequently, for all $j \in \overline{1,n}$ we find, by coupling the chain estimates above, that

$$\int_{Q_{\underline{\omega}_j}} |p_j|^2 e^{2s\overline{\alpha}_j} dxdt \leq C \int_{Q_{\omega_0}} |p_0|^2 e^{2s\overline{\alpha}_0} dxdt, \tag{6.29}$$

which plugged into (6.26) gives a final Carleman estimate

$$\begin{aligned} &\sum_{j \in \overline{0,n}} \int_Q (|D_t p_j|^2 + |D^2 p_j|^2 + |D p_j|^2 + |p_j|^2) e^{2s\underline{\alpha}_j} dxdt \\ &\leq C \int_{Q_{\omega_0}} |p_0|^2 e^{2s\overline{\alpha}_0} dxdt. \end{aligned} \tag{6.30}$$

which gives the final conclusion in the $L^2 - L^2$ framework, (6.20).

The $L^\infty - L^2$ estimate (6.21) follows by the same lines in the corresponding Theorem 1, using the bootstrap argument in connection to the regularity properties of the parabolic flow. ■

The main result concerning controllability with one control for linear parabolic systems with tree-like couplings is the following:

Theorem 5. *Consider system (6.1) with coefficients in $\tilde{\mathcal{E}}_{M,\delta,\{\underline{\omega}_i\}_i}$. Then there exists a constant $C = C(M, \delta, \{\underline{\omega}_i\}_i)$ such that for all $z^0 \in H$ there exists $u^* \in L^2(0,T; L^2(\omega_0)) \cap L^\infty(Q_{\omega_0})$ which drives the corresponding solution to (6.1) in 0, i.e. $z = z^{u^*}$ satisfies $z(T) = 0$ and the control satisfies the norm estimate*

$$\|u^* e^{-s\overline{\alpha}}\|_{L^2(0,T;L^2(\omega_0))} + \|u^*\|_{L^\infty(Q_{\omega_0})} \leq C \|z^0\|_{L^2(\Omega)}. \tag{6.31}$$

Proof. The proof is identical to the proof of Theorem 2 by using the Carleman estimates for the linear adjoint system (6.9) given by Theorem 4 and a corresponding observability estimate as the one given by Remark 2.

Note here that for the L^∞ estimate on the control, one needs to use in Carleman estimate a parameter λ such that (3.16) holds. ■

Controllability of nonlinear semilinear parabolic systems with tree-like couplings may be studied in analogy to the star-like case. For this, consider systems of the form

(6.32)
$$\begin{cases} D_t y_0 - \Delta y_0 = g_0(x) + f_0(x, y_0) + \chi_{\omega_0} u, & \text{in } (0,T)\times\Omega, \\ D_t y_i - \Delta y_i = g_i(x) + f_i(x, y_{\mathbf{k}(i)}, y_i),\ i \in \overline{1,n}, & \text{in } (0,T)\times\Omega, \\ y_0 = ... = y_n = 0, & \text{on } (0,T)\times\partial\Omega, \\ y(0,\cdot) = y^0, \end{cases}$$

where $g_j \in L^\infty(\Omega)$, $j \in \overline{0,n}$ and $\overline{y} = (\overline{y}_0, ..., \overline{y}_n) \in [L^\infty(\Omega)]^{n+1}$ is a corresponding stationary solution. We assume the following hypotheses on the nonlinearities:

(H1') $f_0 \in C^1(\Omega\times\mathbb{R})$, $f_i \in C^1(\Omega\times\mathbb{R}\times\mathbb{R})$, $i \in \overline{1,n}$ there exist $\omega_1, ...\omega_n \subset \Omega$ open nonempty subsets of Ω satisfying (6.4),(6.5) and

$$(\omega_i \cap \omega_{\mathbf{k}(i)}) \setminus \bigcup_{j\neq i, \mathbf{k}(j)=\mathbf{k}(i)} \omega_j \neq \emptyset,\ \forall i \in \overline{1,n}, \tag{6.33}$$

and for all $i \in \overline{1,n}$ we have

$$f_i(x,\tau,\xi) = 0\, \forall x \in \Omega\setminus\omega_i,\ \tau,\xi\in\mathbb{R}; \tag{6.34}$$

(H2') For a family of subdomains $\{\underline{\omega}_i\}_i$ satisfying (6.6),(6.7), by defining for $i \in \overline{1,n}$ the coefficients

$$a^0_{i\mathbf{k}(i)}(x) := \frac{\partial f_i}{\partial y_{\mathbf{k}(i)}}(x, \overline{y}_{\mathbf{k}(i)}(x), \overline{y}_i(x))$$

$$c^0_0(x) := \frac{\partial f_0}{\partial y_0}(x, \overline{y}_0(x)),\ c^0_i(x) := \frac{\partial f_i}{\partial y_i}(x, \overline{y}_{\mathbf{k}(i)}(x), \overline{y}_i(x)),$$

we assume that for some $M_0, \delta_0 > 0$ we have

$$\{a^0_{i\mathbf{k}(i)}, c^0_j\}_{i\in\overline{1,n}, j\in\overline{0,n}} \in \mathcal{E}_{M_0,\delta_0,\{\underline{\omega}_i\}_i,\mathbf{k}}. \tag{6.35}$$

Theorem 6. *Suppose $\overline{y}$ is a stationary state to uncontrolled ($u = 0$) (6.32) and that functions $f_j, j \in \overline{0,n}$ satisfy hypotheses (H1'), (H2'). Then, for all $\beta_0 > 0$ there exist $\zeta_0 = \zeta_0(\beta_0) > 0$ and $C = C(\beta_0, \{\underline{\omega}_i\}_i, \overline{y})$ such that if $\|y^u(0) - \overline{y}\| < \zeta_0$ there exists a control $u \in L^\infty(Q)$ satisfying*

$$\|u\|_{L^\infty(Q)} \le C\|y^u(0) - \overline{y}\|_{L^\infty(\Omega)}$$

and

$$y^u(T,\cdot) = \overline{y},$$

with

$$\|y(t,\cdot) - \overline{y}\|_{L^\infty} \le \beta_0,\ t \in [0,T].$$

Remark 4. (1) *Our results remain valid if instead of the operator Δ we use general elliptic operators wehich may be differently chosen in each of the equation of the system:*

$$L_i y_i := -\sum_{j,k=1}^N D_j(\alpha_i^{jk} D_k y_i) + \sum_{k=1}^N \beta_i^k D_k y_i + \gamma_i y_i \quad i = \overline{1,n}, \tag{6.36}$$

with general boundary conditions which may be also of Neumann or Robin type. Here $(\alpha_i^{jk})_{j,k}$ satisfy uniform ellipticity conditions in Ω. In our study we need also to impose regularity assumptions on the coefficients ($\alpha_i^{jk} \in W^{1,\infty}(\Omega), \beta_i^k, \gamma_i \in L^\infty(\Omega)$); these regularity *assumptions allow the development of the bootstrap argument based on the regularizing properties of the parabolic flow when establishing an* L^∞ *framework for the controllability problem.*

(2) *The hypotheses on the support of the coupling coefficients is essential for our approach to the controllability problem. In fact, for the systems we consider with the same type of couplings but with constant coupling coefficients controllability no longer occurs. take for example the following system with a star-type coupling (α and β are fixed real constants):*

$$\begin{cases} D_t z_0 - \Delta z_0 = \chi_{\omega_0} u, & in\ (0,T)\times\Omega, \\ D_t z_1 - \Delta z_1 = \alpha z_0, & in\ (0,T)\times\Omega, \\ D_t z_2 - \Delta z_2 = \beta z_0, & in\ (0,T)\times\Omega, \\ z_0 = z_1 = z_2 = 0, & on\ (0,T)\times\partial\Omega. \end{cases} \tag{6.37}$$

Considering the results in [1],[2], null controllability occurs if and only if the Kalman rank condition rank$[A_0|B] = 3$. *However, in this situation the Kalman matrix is* $[A_0|B] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & \beta & 0 \end{pmatrix}$ *and its rank is* 2.

Also, if we consider the parabolic system with tree-like couplings (2.13) in §2 Preliminaries, with constant coefficients $c_j = 0$, $a_{10} = a_{20} = a_{31} = a_{41} = 1$, *the Kalman matrix*

$[A_0|B] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$ *and this has rank* 3; *thus the system is not null controllable.*

In fact one may see the results in this paper more as an extension of the results concerning cascade-like parabolic systems with nonconstant coefficients (see [10]).

References

[1] F. Ammar Khodja, A. Benabdallah, C. Dupaix, and M. González-Burgos. A generalization of the Kalman rank condition for time-dependent coupled linear parabolic systems. *Differ. Equ. Appl.*, 1(3):427–457, 2009.

[2] Farid Ammar-Khodja, Assia Benabdallah, Cédric Dupaix, and Manuel González-Burgos. A Kalman rank condition for the localized distributed controllability of a class of linear parbolic systems. *J. Evol. Equ.*, 9(2):267–291, 2009.

[3] Farid Ammar Khodja, Franz Chouly, and Michel Duprez. Partial null controllability of parabolic linear systems. *Math. Control Relat. Fields*, 6(2):185–216, 2016.

[4] V. Barbu. Exact controllability of the superlinear heat equation. *Appl. Math. Optim.*, 42(1):73–89, 2000.

[5] Viorel Barbu. The Carleman inequality for linear parabolic equations in L^q norm. *Differential Integral Equations*, 15(5):513–525, 2002.

[6] Jean-Michel Coron, Sergio Guerrero, and Lionel Rosier. Null controllability of a parabolic system with a cubic coupling term. *SIAM J. Control Optim.*, 48(8):5629–5653, 2010.

[7] Enrique Fernández-Cara and Enrique Zuazua. Controllability for blowing up semilinear parabolic equations. *C. R. Acad. Sci. Paris Sér. I Math.*, 330(3):199–204, 2000.

[8] Enrique Fernández-Cara and Enrique Zuazua. The cost of approximate controllability for heat equations: The linear case. *Adv. Differ. Equ.*, 5(4-6):465–514, 2000.

[9] A.V. Fursikov and O.Yu. Imanuvilov. *Controllability of evolution equations.* Seoul: Seoul National Univ., 1996.

[10] Manuel González-Burgos and Luz de Teresa. Controllability results for cascade systems of m coupled parabolic PDEs by one control force. *Port. Math.*, 67(1):91–113, 2010.

[11] O. A. Ladyzenskaja, V. A. Solonnikov, and N. N. Uralceva. *Linear and quasilinear equations of parabolic type.* Vol. 23. American Mathematical Society, Providence, R.I., 1968.

[12] Kévin Le Balc'h. Controllability of a 4×4 quadratic reaction-diffusion system. *J. Differential Equations*, 266(6):3100–3188, 2019.

[13] Pierre Lissy and Enrique Zuazua. Internal observability for coupled systems of linear partial differential equations. *SIAM J. Control Optim.*, 57(2):832–853, 2019.

[14] Guillaume Olive. Null-controllability for some linear parabolic systems with controls acting on different parts of the domain and its boundary. *Math. Control Signals Systems*, 23(4):257–280, 2012.

[15] Ioan I. Vrabie. *C_0-semigroups and applications*, volume 191 of *North-Holland Mathematics Studies.* North-Holland Publishing Co., Amsterdam, 2003.

(C.G. Lefter, E.A. Melnig) Octav Mayer Institute of Mathematics, Romanian Academy, Iaşi Branch and Faculty of Mathematics, University "Al. I. Cuza" Iaşi, Romania

E-mail address: catalin.lefter@uaic.ro, alex.melnig@yahoo.com