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The isotropic Cosserat shell model including terms up  
to  $O(h^5)$   
**Part I: Derivation in matrix notation**

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# The isotropic Cosserat shell model including terms up to $O(h^5)$

## Part I: Derivation in matrix notation

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### Abstract

We present a new geometrically nonlinear Cosserat shell model incorporating effects up to order  $O(h^5)$  in the shell thickness  $h$ . The method that we follow is an educated 8-parameter ansatz for the three-dimensional elastic shell deformation with attendant analytical thickness integration, which leads us to obtain completely two-dimensional sets of equations in variational form. We give an explicit form of the curvature energy using the orthogonal Cartan-decomposition of the wryness tensor. Moreover, we consider the matrix representation of all tensors in the derivation of the variational formulation, because this is convenient when the problem of existence is considered, and it is also preferential for numerical simulations. The step by step construction allows us to give a transparent approximation of the three-dimensional parental problem. The resulting 6-parameter isotropic shell model combines membrane, bending and curvature effects at the same time. The Cosserat shell model naturally includes a frame of orthogonal directors, the last of which does not necessarily coincide with the normal of the surface. This rotation-field is coupled to the shell-deformation and augments the well-known Reissner-Mindlin kinematics (one independent director) with so-called in-plane drill rotations, the inclusion of which is decisive for subsequent numerical treatment and existence proofs. As a major novelty, we determine the constitutive coefficients of the Cosserat shell model in dependence on the geometry of the shell which are otherwise difficult to guess.

**Keywords:** geometrically nonlinear Cosserat shell, 6-parameter resultant shell, in-plane drill rotations, thin structures, dimensional reduction, Cosserat elasticity, wryness tensor, dislocation density tensor, isotropy

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# 1 Introduction

The theory of shells is an important branch of the theory of deformable solids. Its importance resides in the multitude of applications that can be investigated using shell models. In general, the shell and plate theories are intended for the study of *thin bodies*, i.e. bodies in which the thickness in one direction is much smaller than the dimensions in the other two orthogonal directions. As typical examples for shells we mention: roofs of buildings in civil engineering, vehicle bodies in automotive industry, components of wings and propellers in aerospace industry, cell walls and biological membranes.

The large variety of shell-type structures, such as laminated or functionally graded shells made of advanced materials, like polymer foams or cellular materials, as well as the need to fabricate three-dimensional micro- and nanostructures of various shapes leads to the necessity of elaborating new adequate models to describe their mechanical behavior. The process of development of various shell theories is far from being finalized, due to the continuous emergence of new technologies in connection with shell manufacturing. For instance, the emerging need to simulate the mechanical response of highly flexible ultra thin structures (allowing easily for finite rotations) and nano-scale thin structures excludes the use of classical infinitesimal–displacement models, either of Reissner–Mindlin or Kirchhoff–Love type. In particular, graphene sheets consisting of monolayer atomic arrangement have non-vanishing “bending stiffness”. This is at odds with classical thin shell theory, in which, for  $h \rightarrow 0$ , the bending stiffness should be absent (since the bending terms scale with  $h^3$  against  $h$  for the membrane strains). Instead, in an extended continuum model like the *Cosserat-type shell model*, there is a “curvature stiffness” surviving for  $h \rightarrow 0$  related to a characteristic size  $L_c > 0$  (internal length parameter, which is related to the microstructure and in principle independent of the thickness).

## 1.1 Different approaches to shell modelling

There are several alternative ways to describe the mechanical behavior of shells and to derive the two-dimensional field equations. The method used in our paper is the so-called *derivation approach*. It starts from a given three-dimensional model of the body and reduces it via physically reasonable constitutive assumptions on the kinematics to a two-dimensional model (i.e. *dimensional reduction*). The philosophy behind the derivation approach is expressed clearly by the grandmaster W.T. Koiter as follows [50, p. 93]: “Any two-dimensional theory of thin shells is necessarily of an *approximate character*. An exact two-dimensional theory of shells cannot exist, because the actual body we have to deal with, thin as it may be, is always three-dimensional. [...] Since the theory we have to deal with is approximate in character, we feel that extreme rigour in its development is hardly desirable. [...] *Flexible bodies* like thin shells require a *flexible approach*.” We mention also that the

“rationale” of descend from three to two dimensions should always be complemented by an investigation of the intrinsic mathematical properties of the lower dimensional models.

This procedure is opposed to both the *intrinsic approach* and the *asymptotic methods*. The intrinsic approach is related to the *derivation approach* but it takes the shell a priori to be a two-dimensional surface (appropriate for modelling graphene sheets, with virtually zero thickness) with additional extrinsic directors in the sense of a deformable Cosserat surface [31, 32, 1]. There, two-dimensional equilibrium in appropriate resultant stress and strain variables is postulated ab-initio more or less independently of three-dimensional considerations. For instance, in the shell model elaborated by Zhilin and Altenbach [98, 2] each point of the surface is endowed with a triad of orthonormal vectors (called *directors*), which specify the orientation of the material points and describe the *rotations* of shell filaments. Several interesting applications of this approach (also called *directed surfaces*) have been presented in the work [5]. In this approach, the constitutive parameters appearing in the a priori chosen two-dimensional energy are obtained by fitting the solutions of some specific problems with the solutions obtained by considering the shell as a three-dimensional body. However, this fitting is usually done only for linear problems, since one reason of being of the nonlinear shell model is that even the classical three-dimensional problem is difficult to be solved for nonlinear problems. Therefore, it is still missing a complete description of the dependence of the constitutive parameters of the reduced nonlinear two-dimensional problem on the constitutive parameters of the parental nonlinear three-dimensional problem and on the mean-curvature  $H$  and Gauß-curvature  $K$  of the shell’s initially stress-free midsurface. It is possible that using the intrinsic approach, certain three-dimensional effects may be missing in the derived shell model. Another approach which is also related to the derivation approach is the *uniform-approximation technique*, mostly motivated by engineering intuition [95, 94, 12, 46, 47, 4, 86]. It uses polynomial expansions in the thickness direction both for the displacements and for the stresses and then it truncates the series expansions.

The procedure of the *asymptotic methods* is to establish two-dimensional equations by formal expansion of the three-dimensional solutions in power series in terms of a small thickness parameter. Using the asymptotic methods, a thorough mathematical analysis of linear, infinitesimal displacement shell theory is presented in [22] and the extensive references therein. Properly invariant, geometrically nonlinear elastic shell theories are derived by formal asymptotic methods in [54, 52], see also [17]. The various shell models based on linearized three-dimensional elasticity proposed in the literature have been rigorously justified in those cases, where some normality assumption is introduced, either a priori or as a result of an asymptotic analysis, see notably the extensive work of Ciarlet and his collaborators [22, 27], see also [89, 91, 90]. However, even in the infinitesimal-displacement case it becomes apparent that a model involving membrane and bending simultaneously, cannot be obtained by formal asymptotic methods. In some landmark contributions [34, 41], see also [40, 42] based on the  $\Gamma$ -limit of the three-dimensional model for vanishing thickness, the “membrane-dominated model” and the “flexural-dominated model”, respectively, are obtained. The difference between asymptotic methods and  $\Gamma$ -convergence is that the methods based on  $\Gamma$ -convergence [34] lead to another finite-strain membrane term, indicating a nonresistance of the membrane shell in compression. In some examples of careful modelling, the (derivation) Koiter model [49, 88], is simply the sum of the correctly identified membrane and bending contributions, properly scaled with the thickness (the membrane terms scale with  $h$  and bending terms with  $h^3$ ). It is shown in [26] that the Koiter model is asymptotically at least as good as either the membrane model or the bending model in the respective deformation regimes. Regarding the bending term, agreement has been reached that the term which is consistent with the three-dimensional isotropic Saint-Venant–Kirchhoff energy

$$W_{\text{SVK}}(F) = \frac{\mu}{4} \|U^2 - \mathbb{1}_3\|^2 + \frac{\lambda}{8} [\text{tr}(U^2 - \mathbb{1}_3)]^2 = \frac{\mu}{4} \|C - \mathbb{1}_3\|^2 + \frac{\lambda}{8} [\text{tr}(C - \mathbb{1}_3)]^2, \quad (1.1)$$

where  $C = U^2 = F^T F$ , is a quadratic expression in the second fundamental form  $\Pi_{y_0}$  of the surface. Nevertheless, the model obtained by energy projection in [42] differs from the results obtained by formal asymptotic analysis in [40]. For the finite-strain membrane model, no rigorous justification of the formal asymptotic approach has been given, because of the lack of a theory which guarantees the well-posedness of the nonlinear three-dimensional problem based on (1.1). Since the membrane terms in a finite-strain properly invariant Kirchhoff-Love shell or finite-strain Reissner-Mindlin model are non-elliptic, the remaining minimization problem is not well-posed even if classical bending is present. By contrast, this is not the case, when a nonlinear three-dimensional problem in the Cosserat theory is considered [65, 93, 92]. By ignoring the Cosserat effects, we will be forced to consider a polyconvex energy [3] in the three-dimensional formulation of the initial problem. These energies do not allow an easy manipulation with respect to the approaches described in this subsection.

In this direction, an example is the article [29], see also [25, 15], where the Ciarlet-Geymonat energy [24] is used. In these articles, no reduced completely two-dimensional minimization problem is presented and no through the thickness integration is performed analytically. The obtained problems are “two-dimensional” only in the sense that the final problem is to find three vector fields defined on a bounded open subset of  $\mathbb{R}^2$ , but all three-dimensional coordinates remain present in the minimization problem. Moreover, the obtained minimization problem is compared with the Koiter model only for small strain-tensors, situation in which the considered energy is actually the isotropic Saint-Venant–Kirchhoff energy.

In applications there are usually regions of a shell where membrane effects dominate, while bending is dominant in others, but both have to be present in the general model. A fully three-dimensional resolution of a thin shell problem remains elusive at present, notwithstanding the increase in computing power. Hence, there is still a need to come up with a sound finite-strain shell model, combining both effects of membrane and bending in one system of equations, as the Koiter model does successfully in the infinitesimal-displacement case.

There are numerous proposals in the engineering literature for such a finite-strain, geometrically nonlinear shell formulation. In many cases, the need has been felt to devote attention to the rotation field  $\bar{R} \in \text{SO}(3)$ , since rotations are locally the dominant deformation mode of a thin flexible structure. We also mention that considering the new unknown field  $\bar{R} \in \text{SO}(3)$ , we are able to keep more three-dimensional effects into our two-dimensional variational formulation (local independent rotations in various directions, i.e. each material point of the body has the degrees of freedom of a rigid body), in contrast to the shell models based on classical elasticity. In fact, this is one of the first *raison d’être* of the Cosserat theory [31, 32]. This has led to shell models which include the so-called *drilling rotations* [97], meaning that in-plane rotations about the shell filament are also taken into account. In [87] it is shown that the inclusion of drilling rotations in the model has a beneficial influence on the numerical implementation. However, a mathematical analysis for such a family of finite-strain curved shell models is, as of yet, still missing.

One of the most general and effective approaches to shells is the so called geometrically nonlinear 6-parameter resultant shell theory, which was originally proposed by Reissner [79] and has subsequently been presented considerably. An account of these developments and main achievements have been presented in the works of Libai and Simmonds [51] and Pietraszkiewicz et al. [76, 36]. This model involves two independent kinematic fields: the translation vector field and the rotation tensor field (in total six independent scalar kinematic variables). The two-dimensional equilibrium equations and static boundary conditions of the shell are derived exactly by direct through-the-thickness integration of the stresses in the three-dimensional balance laws of linear and angular momentum. The kinematic fields are then constructed directly on the two-dimensional level using the integral identity of the virtual work principle. Following this procedure, the two-dimensional model is expressed in terms of stress resultants and work-averaged deformation fields defined on the shell base surface. The kinematical structure of this 6-parameter model (involving the translation vector and rotation tensor) is identical to the kinematical structure of Cosserat shells (defined as material surfaces endowed with a triad of rigid directors describing the orientation of points). Several developments of this model and applications to complex shell problems, including phase transition and multifold shells, together with the finite element implementation, have been presented by Pietraszkiewicz, Eremeyev and Chróścielewski with their co-workers in a number of papers [36, 37, 19, 76, 16], see also [77].

## 1.2 The new shell model presented in this article

In this paper, we extend the Cosserat plate model established by Neff in his habilitation thesis in 2003 [58, 63, 96, 60, 61] to the general case of curved initial shell configurations. The results have been previously announced (using a succinct tensor notation) in [6, 64] and the aim of the current article is to explain the derivation of the model in more details, with added transparency, and using only the matrix representation. Moreover, all our calculations do not use curvilinear coordinates and do not use an a priori parametrization of the mid-surface, since at the very beginning we are starting by considering the general form of the three-dimensional deformation energy of a fully three-dimensional body, without involving the informations about its mid-surface in the variational formulation. To be more precise, our initial constitutive assumptions (the form of the energy) do not, of course, depend on the shape of the three-dimensional body.

We start with a suitable bulk three-dimensional isotropic Cosserat model written in Cartesian coordinates, we make an appropriate ansatz for the deformation and rotation functions and we perform the integration over the thickness. The form we chose for the three-dimensional Cosserat energy is already a strong point of our

new approach, since it will help us to give an explicit analytic form of the entire energy of the shell model, and therefore it will be useful in analytical and numerical studies. More precisely, we give an explicit form of the curvature energy using the orthogonal Cartan-decomposition of the wryness tensor (the used curvature tensor of the Cosserat bulk model). In the modelling process we follow the derivation approach as described for planar configurations in [58], but we need to additionally incorporate the curvature effects by using known results from the differential geometry of surfaces in  $\mathbb{R}^3$ . Thus, we obtain a geometrically nonlinear formulation for Cosserat-type shells with 6 independent kinematical variables: 3 for displacements and 3 for finite rotations (including drilling rotations).

The new model will resolve some shortcomings of classical approaches, which we have mentioned in the previous subsections. In particular, it satisfies the following requirements which we deem necessary for an effective general shell model:

- a geometrically nonlinear formulation that allows for finite rotations.
- the description of transverse shear, drilling rotations, thickness stretch and asymmetric shift of the mid-surface.
- a hyperelastic, variational formulation with second-order Euler-Lagrange equations in view of an efficient finite element implementation.
- a dimensionally reduced energy density which is entirely defined in terms of two-dimensional quantities with a clear physical meaning and by a step by step construction.
- well-posedness: existence of solutions, but not unqualified uniqueness in order to be able to describe buckling due to membrane forces (e.g., under lateral compression).
- the consistency with classical shell models for infinitesimal deformations.
- the incorporation of non-classical size effects, such that graphene sheets have bending/curvature resistance.

Since we begin with a Cosserat bulk model which already contains in its formulation the so-called *Cosserat couple modulus*  $\mu_c \geq 0$  and an internal length  $L_c > 0$  (which is characteristic for the material, e.g. related to the grain size in a polycrystal), the reduced energy density for shells will also include the material parameters  $\mu_c$  and  $L_c$ , in conjunction with specific terms having a clear physical meaning, expressed as functions of two-dimensional quantities. The internal length parameter  $L_c$  allows for the incorporation of non-classical size effects in the shell model (in the sense that smaller samples are relatively stiffer than larger samples [30]).

For very irregular and curved initial shell configurations it is not at all clear which terms get small in a thin shell approximation. Moreover, when another dimension (beside thickness) of the parental three-dimensional body is very small or when the deformations are large compared to the thickness, terms of order  $O(h^3)$  may not be sufficient to capture important three-dimensional behaviour. Therefore, we aim to elaborate a complete and consistent model up to the order  $O(h^5)$ , i.e. we determine all the terms up to the order  $O(h^5)$  in both the membrane part and the bending-curvature part of the energy density. Indeed, in a shell model  $h$  is very small, but the reason of being of a shell model is to obtain an approximation of the deformation of a three-dimensional body. By considering terms up to the order  $O(h^5)$ , some additional three-dimensional effects are not omitted in the obtained two-dimensional model. The used method allows this construction in a very transparent way without considering at the very beginning a two-dimensional problem, approach in which terms of order  $O(h^5)$  cannot be guessed a priori. Moreover, the coefficients of the terms in the energy density depend on the mean-curvature  $H$  and Gauß-curvature  $K$  of the shell's midsurface, the calculations showing us that these coefficients have unforeseeable expressions. Thus, we come up with an improved model which should generalize most of the known variants of shell theory (since they consider only terms of order  $O(h^3)$ , see e.g. [88]). In this respect we will deliver more accurate qualitative and numerical results in forthcoming papers.

We regard also other shell models from the literature as particular cases of our formulation. For instance, the 6-parameter resultant shell theory [36] can be viewed as a special case, since it is a theory of order  $O(h^3)$ , it omits all mixed terms, it is not elaborated starting from a three-dimensional parental problem, and their constitutive coefficients are not expressed in terms of the mean-curvature and Gauß-curvature and not in terms of the constitutive coefficients of the three-dimensional internal energy. In a forthcoming paper, we will show that our extended model is consistent with the classical linear Koiter model, in view of Ciarlet's remark: "The stored energy function of any realistic shell model should coincide to 'within first order' with the classical Koiter shell model"—at the 8<sup>th</sup> European Solid Mechanics Conference (EUROMECH, Graz, July 9-13, 2012).

The present paper is completely self-contained and can be read also by researchers not yet accustomed to the specific notation and usages of shell-theory. We arrive at this point at the expense of working, as far as

possible, with concepts from 3D-elasticity theory as well as consequently utilizing “reconstructed” 3D-matrix objects. Thus the paper is ideally suited for researchers in need of quickly understanding the basic ingredients of a geometrically nonlinear shell theory. In forthcoming papers we will compare a suitable restriction of our modelling framework with the geometrically nonlinear  $O(h^3)$ -Koiter model. Preliminary observations suggests that our model (restricted to the same order  $O(h^3)$ ) includes terms not present in the Koiter model (isotropic Kirchhoff-Love shell). For developable surfaces (Gauß-curvature  $K = 0$ ) and after linearisation, both approaches seem to coincide. We will also investigate a corresponding  $\Gamma$ -convergence result, similar to [69, 68, 66].

The considered matrix representation, in the entire derivation of the model, is more convenient when the problem of existence is considered, it is also preferred for numerical simulations in the engineering community and it offers some details about how the elastic shell model obtained in the present article may be extended to an elasto-plastic shell model [57, 59] or to a model which includes residual (initial) stresses [53]. These subjects will be considered in future works based on our model. These types of thin bodies are of great importance nowadays, in view of the new shell manufacturing procedures and we believe that the terms of order  $O(h^5)$  included here will play important roles in increasing the accuracy of analytical and numerical predictions in these industrial processes. In forthcoming papers we will prove the existence of the solution of the obtained minimization problem [43] (at order  $O(h^3)$  and  $O(h^5)$ ), we will offer some numerical simulations similar to [82] in order to compare our model with some other previous models from literature. Moreover, the pure elastic nonlinear Cosserat shell model will also be extended to viscoelasticity and multiplicative plasticity [60, 84, 81] and it will allow us to discuss residual stress effects in applications to design-control problems of nano-three-dimensional objects [33], situations in which a model up to order  $O(h^5)$  is useful since another dimension (beside thickness) may be very small or the deformations are large compared to thickness. Therefore, a model up to order  $O(h^5)$  may be very useful in the study of thin bodies with a relative not “so small” thickness compared to the other two dimensions, e.g. in the study of nano-structures.

## 2 The three-dimensional formulation

In [58] a physically linear, fully frame-invariant isotropic Cosserat model is introduced. The problem has been posed in a variational setting. We consider a three-dimensional *shell-like thin domain*  $\Omega_\xi \subset \mathbb{R}^3$ . A generic point of  $\Omega_\xi$  will be denoted by  $(\xi_1, \xi_2, \xi_3)$  in a fixed standard base vector  $e_1, e_2, e_3$  of  $\mathbb{R}^3$ . The elastic material constituting the shell is assumed to be homogeneous and isotropic and the reference configuration  $\Omega_\xi$  is assumed to be a natural (stress-free) state. In the rest of the present article we use the notation given in Appendix A.1.

The deformation of the body occupying the domain  $\Omega_\xi$  is described by a map  $\varphi_\xi$  (*called deformation*) and by a *microrotation*  $\bar{R}_\xi$ ,

$$\varphi_\xi : \Omega_\xi \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad \bar{R}_\xi : \Omega_\xi \subset \mathbb{R}^3 \rightarrow \text{SO}(3). \quad (2.1)$$

We denote the current configuration by  $\Omega_c := \varphi_\xi(\Omega_\xi) \subset \mathbb{R}^3$ . The deformation and the microrotation is solution of the following *geometrically nonlinear minimization problem* posed on  $\Omega_\xi$ :

$$I(\varphi_\xi, F_\xi, \bar{R}_\xi, \alpha_\xi) = \int_{\Omega_\xi} [W_{\text{mp}}(\bar{U}_\xi) + W_{\text{curv}}(\alpha_\xi)] dV(\xi) - \Pi(\varphi_\xi, \bar{R}_\xi) \rightarrow \min. \quad \text{w.r.t.} \quad (\varphi_\xi, \bar{R}_\xi), \quad (2.2)$$

where

$$\begin{aligned} F_\xi &:= \nabla_\xi \varphi_\xi \in \mathbb{R}^{3 \times 3} && \text{(the deformation gradient),} \\ \bar{U}_\xi &:= \bar{R}_\xi^T F_\xi \in \mathbb{R}^{3 \times 3} && \text{(the non-symmetric Biot-type stretch tensor),} \\ \alpha_\xi &:= \bar{R}_\xi^T \text{Curl}_\xi \bar{R}_\xi \in \mathbb{R}^{3 \times 3} && \text{(the second order dislocation density tensor [71]),} \end{aligned} \quad (2.3)$$

$$W_{\text{mp}}(\bar{U}_\xi) := \mu \|\text{dev sym}(\bar{U}_\xi - \mathbf{1}_3)\|^2 + \mu_c \|\text{skew}(\bar{U}_\xi - \mathbf{1}_3)\|^2 + \frac{\kappa}{2} [\text{tr}(\text{sym}(\bar{U}_\xi - \mathbf{1}_3))]^2 \quad \text{(physically linear),}$$

$$W_{\text{curv}}(\alpha_\xi) := \mu L_c^2 (b_1 \|\text{dev sym} \alpha_\xi\|^2 + b_2 \|\text{skew} \alpha_\xi\|^2 + b_3 [\text{tr}(\alpha_\xi)]^2) \quad \text{(curvature energy),}$$

and  $dV(\xi)$  denotes the volume element in the  $\Omega_\xi$ -configuration.

The total elastically stored energy  $W = W_{\text{mp}} + W_{\text{curv}}$  depends on the deformation gradient  $F_\xi$  and microrotations  $\bar{R}_\xi$  together with their spatial derivatives. In general, the *Biot-type stretch tensor*  $\bar{U}_\xi$  is not symmetric (the first Cosserat deformation tensor [32]). The parameters  $\mu$  and  $\lambda$  are the *Lamé constants* of classical isotropic

elasticity,  $\kappa = \frac{2\mu+3\lambda}{3}$  is the *infinitesimal bulk modulus*,  $b_1, b_2, b_3$  are *non-dimensional constitutive curvature coefficients (weights)*,  $\mu_c \geq 0$  is called the *Cosserat couple modulus* and  $L_c > 0$  introduces an *internal length* which is characteristic for the material, e.g. related to the grain size in a polycrystal. The internal length  $L_c > 0$  is responsible for *size effects* in the sense that smaller samples are relatively stiffer than larger samples. If not stated otherwise, we assume that  $\mu > 0$ ,  $\kappa > 0$ ,  $\mu_c > 0$ ,  $b_1 > 0$ ,  $b_2 > 0$ ,  $b_3 > 0$ .

The form of the curvature energy  $W_{\text{curv}}$  is not that originally considered in [58]. Indeed, Neff [58] uses a curvature energy expressed in terms of the *third order curvature tensor*  $\mathfrak{K} = (\overline{R}_\xi^T \nabla(\overline{R}_\xi \cdot e_1) | \overline{R}_\xi^T \nabla(\overline{R}_\xi \cdot e_2) | \overline{R}_\xi^T \nabla(\overline{R}_\xi \cdot e_3))$ . As we will remark in Section 3, the new form of the energy based on the *second order dislocation density tensor*  $\alpha_\xi$  simplifies considerably the representation by admitting to use the orthogonal decomposition

$$\overline{R}_\xi^T \text{Curl}_\xi \overline{R}_\xi = \alpha_\xi = \text{dev sym } \alpha_\xi + \text{skew } \alpha_\xi + \frac{1}{3} \text{tr}(\alpha_\xi) \mathbb{1}_3. \quad (2.4)$$

Moreover, it yields an equivalent control of spatial derivatives of rotations [71] and allows us to write the curvature energy in a fictitious Cartesian configuration in terms of the so-called *wryness tensor*. This fact has some further implications, e.g. the coupling between the membrane part, the membrane-bending part, the bending-curvature part and the curvature part of the energy of the shell model is transparent and will coincide with shell-bending curvature tensors elsewhere considered [36].

In (2.2),  $\Pi(\varphi_\xi, \overline{R}_\xi)$  is the external loading potential, which admits the following additive decomposition:

$$\Pi(\varphi_\xi, \overline{R}_\xi) = \Pi_f(\varphi_\xi) + \Pi_t(\varphi_\xi) + \Pi_\Omega(\overline{R}_\xi) + \Pi_{\partial\Omega_t}(\overline{R}_\xi), \quad (2.5)$$

where

$$\begin{aligned} \Pi_f(\varphi_\xi) &:= \int_{\Omega_\xi} \langle f, u \rangle dV(\xi) = \text{potential of external applied body forces } f, \\ \Pi_t(\varphi_\xi) &:= \int_{\partial\Omega_t} \langle t, u \rangle dS(\xi) = \text{potential of external applied boundary forces } t, \\ \Pi_\Omega(\overline{R}_\xi) &:= \text{potential of external applied body couples,} \\ \Pi_{\partial\Omega_t}(\overline{R}_\xi) &:= \text{potential of external applied boundary couples,} \end{aligned} \quad (2.6)$$

and  $u = \varphi_\xi - \xi$  is the displacement vector,  $\partial\Omega_t$  is a subset of the boundary of  $\Omega_\xi$ , and  $dS(\xi)$  is the area element.

## 2.1 Relation to the Biot nonlinear elasticity model

The used three-dimensional Cosserat model can be seen as an extension of the geometrically nonlinear isotropic Biot-model. Indeed, letting formally<sup>1</sup>  $\mu_c \rightarrow +\infty$  and  $L_c \rightarrow 0$ , the independent rotation field  $\overline{R} \rightarrow \text{polar}(F)$  must coincide with the continuum rotation in the polar decomposition of  $F = RU = \text{polar}(F) \sqrt{F^T F}$ . Since for  $L_c \rightarrow 0$  curvature is absent, the resulting minimization problem is

$$\int_{\Omega_\xi} [W_{\text{Biot}}(F) - \langle f, \varphi \rangle] dV(\xi) \rightarrow \min. \quad \text{w.r.t. } (\varphi_\xi), \quad (2.7)$$

where

$$W_{\text{Biot}}(F) = \mu \|U - \mathbb{1}_3\|^2 + \frac{\lambda}{2} [\text{tr}(U - \mathbb{1}_3)]^2 = \mu \|\text{dev sym}(U - \mathbb{1}_3)\|^2 + \frac{\kappa}{2} [\text{tr}(\text{sym}(U - \mathbb{1}_3))]^2. \quad (2.8)$$

Recall that typically, the Koiter shell-model is obtained based on the dimension reduction from the isotropic Saint-Venant–Kirchhoff energy

$$W_{\text{SVK}}(F) = \frac{\mu}{4} \|U^2 - \mathbb{1}_3\|^2 + \frac{\lambda}{8} [\text{tr}(U^2 - \mathbb{1}_3)]^2 = \frac{\mu}{4} \|C - \mathbb{1}_3\|^2 + \frac{\lambda}{8} [\text{tr}(C - \mathbb{1}_3)]^2, \quad (2.9)$$

where  $C = U^2 = F^T F$ . Both energies (2.8) and (2.9) are linearisation-equivalent and meant to well-capture the small strain regime expected for the response of a thin shell. However,  $W_{\text{SVK}}(F)$  introduces physically

<sup>1</sup>In order to arrive at the limit Biot model for  $\lambda = 0$ , it is sufficient to consider  $L_c \rightarrow 0$  and  $\mu_c \geq \mu$ , see [38, 39, 67, 11].



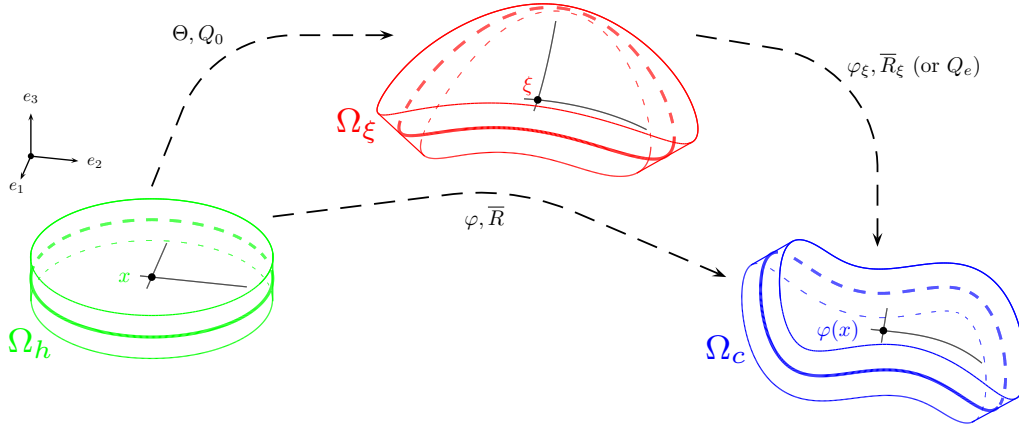


Figure 1: The shell in its initial configuration  $\Omega_\xi$ , the shell in the deformed configuration  $\Omega_c$ , and the fictitious planar cartesian reference configuration  $\Omega_h$ . Here,  $\bar{R}_\xi$  is the elastic rotation field,  $Q_0$  is the initial rotation from the fictitious planar cartesian reference configuration to the initial configuration  $\Omega_\xi$ , and  $\bar{R}$  is the total rotation field from the fictitious planar cartesian reference configuration to the deformed configuration  $\Omega_c$ .

unacceptable behaviour under the slightest compression (compression will be softer than tension). Since  $W_{\text{Biot}}$  does not have this feature, we believe that arriving with our model at  $W_{\text{Biot}}$  is advantageous.

Preliminary calculations show us that, in some particular cases, the total energy of the Cosserat-shell model constructed by using the Biot energy reduces to quadratic and bilinear forms in terms of the difference of the squares of the first fundamental forms (of the initial configuration and of the current configuration) and/or in terms of the difference of the second fundamental forms. This is in concordance to the new estimates of the distance between two surfaces obtained in [28, 23] which anticipate the need for this type of energies.

### 3 Transformed variational problem in the fictitious configuration $\Omega_h$

In what follows, we assume that the parameter domain  $\Omega_h \subset \mathbb{R}^3$  is a right cylinder of the form

$$\Omega_h = \left\{ (x_1, x_2, x_3) \mid (x_1, x_2) \in \omega, -\frac{h}{2} < x_3 < \frac{h}{2} \right\} = \omega \times \left( -\frac{h}{2}, \frac{h}{2} \right),$$

where  $\omega \subset \mathbb{R}^2$  is a bounded domain with Lipschitz boundary  $\partial\omega$  and the constant length  $h > 0$  is the *thickness of the shell*. For shell-like bodies we consider the domain  $\Omega_h$  to be thin, i.e. the thickness  $h$  is small. Thus, the domain  $\Omega_h$  can be viewed as a *fictitious Cartesian configuration* of the body.

We assume furthermore that there exists a given  $C^1$ -diffeomorphism  $\Theta : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ , which maps the fictitious Cartesian parameter space  $\Omega_h$  with coordinates  $(x_1, x_2, x_3) \in \mathbb{R}^3$  onto  $\Theta(x_1, x_2, x_3) = (\xi_1, \xi_2, \xi_3)$  such that the initially curved reference configuration of the shell is  $\Theta(\Omega_h) = \Omega_\xi$  (see Figure 1).

Now, let us define the map

$$\varphi : \Omega_h \rightarrow \Omega_c, \quad \varphi(x_1, x_2, x_3) = \varphi_\xi(\Theta(x_1, x_2, x_3)). \quad (3.1)$$

We view  $\varphi$  as a function which maps the fictitious planar reference configuration  $\Omega_h$  into the deformed (current) configuration  $\Omega_c$ . Hence, the guiding question is: how can we construct the map  $\varphi$  and a total rotation tensor  $\bar{R}$  in order to reduce suitably the three-dimensional problem to a two-dimensional problem? To answer this question (see Figure 1) we reformulate the minimization principle (2.2) in the fictitious, Cartesian configuration  $\Omega_h$ . If we construct such mappings, since the diffeomorphism  $\Theta$  is considered known, then we also know the map which describes the deformation of the initial curved reference configuration  $\Omega_\xi$  into the current configuration  $\Omega_c$  of the body.

Assume an underlying three-dimensional deformation of the shell-like body  $\varphi_\xi$  is known and differentiable. Consider a point  $\beta = (x_1, x_2, 0) \in \omega \times \{0\}$  and  $\Theta(\beta)$ . For the moment, we do not assume that  $\Theta(\beta)$  is mapped

to the midsurface of  $\Omega_\xi = \Theta(\Omega_h)$ . Consider also the point  $\beta_{x_3} = (x_1, x_2, x_3)$ , i.e. the line  $\beta\beta_{x_3}$  is normal to  $\omega$ . Then, we have the expansion

$$\begin{aligned} \varphi_\xi(\Theta(\beta_{x_3})) &= \varphi_\xi(\Theta(\beta)) + x_3 \nabla_\xi \varphi_\xi(\Theta(\beta)) \nabla_x \Theta(\beta) \cdot e_3 + o(x_3) && \text{or} \\ \varphi_\xi(\Theta(x_1, x_2, x_3)) &= \underbrace{\varphi_\xi(\Theta(x_1, x_2, 0))}_{:=m_\xi} + x_3 \nabla_\xi \varphi_\xi(\Theta(x_1, x_2, 0)) \nabla_x \Theta(x_1, x_2, 0) \cdot e_3 + o(x_3), \end{aligned} \quad (3.2)$$

where  $\Theta(x_1, x_2, 0)$  does not belong to the midsurface of  $\Omega_\xi$ , but to the transformed midsurface  $\omega_\xi = \Theta(\omega \times \{0\})$ , as long as  $\Theta$  has not a specific expression.

### 3.1 Transformation of the minimization problem

Consider the *elastic microrotation*

$$\bar{Q}_e : \Omega_h \rightarrow \text{SO}(3), \quad \bar{Q}_e(x_1, x_2, x_3) := \bar{R}_\xi(\Theta(x_1, x_2, x_3)) \quad (3.3)$$

and the *elastic (non-symmetric) Biot-type stretch tensor* (the elastic first Cosserat deformation tensor)

$$\bar{U}_e : \Omega_h \rightarrow \text{Sym}(3), \quad \bar{U}_e(x_1, x_2, x_3) := \bar{U}_\xi(\Theta(x_1, x_2, x_3)). \quad (3.4)$$

We use the polar decomposition [70] of  $\nabla_x \Theta$  and write

$$\nabla_x \Theta = Q_0 U_0, \quad Q_0 = \text{polar}(\nabla_x \Theta) = \text{polar}([\nabla_x \Theta]^{-T}) \in \text{SO}(3), \quad U_0 \in \text{Sym}^+(3). \quad (3.5)$$

Corresponding to the elastic deformation process, we have the total microrotation

$$\bar{R} : \Omega_h \rightarrow \text{SO}(3), \quad \bar{R}(x_1, x_2, x_3) = \bar{Q}_e(x_1, x_2, x_3) Q_0(x_1, x_2, x_3). \quad (3.6)$$

Obviously, if we know the total microrotation  $\bar{R}$ , then we know the microrotation  $\bar{R}_\xi$ . Using the chain rule

$$\partial_{x_k} \varphi = \sum_{i=1}^3 \partial_{\xi_i} \varphi_\xi \partial_{x_k} \xi_i, \quad \nabla_x \varphi(x_1, x_2, x_3) = \nabla_\xi \varphi_\xi(\Theta(x_1, x_2, x_3)) \nabla_x \Theta(x_1, x_2, x_3), \quad (3.7)$$

we deduce (the multiplicative decomposition)

$$\begin{aligned} F(x_1, x_2, x_3) &= F_\xi(\Theta(x_1, x_2, x_3)) \nabla_x \Theta(x_1, x_2, x_3), && \text{where } F = \nabla_x \varphi, \\ F_\xi(\Theta(x_1, x_2, x_3)) &= F(x_1, x_2, x_3) [\nabla_x \Theta(x_1, x_2, x_3)]^{-1}. \end{aligned} \quad (3.8)$$

Therefore, the elastic non-symmetric stretch tensor is given by

$$\bar{U}_e = \bar{Q}_e^T F [\nabla_x \Theta]^{-1} = Q_0 \bar{R}^T F [\nabla_x \Theta]^{-1}. \quad (3.9)$$

As a Lagrangian strain measure for curvature (orientation change) one can also employ the so-called *wryness tensor* (second order tensor) [71, 35]

$$\Gamma_\xi := \left( \text{axl}(\bar{R}_\xi^T \partial_{\xi_1} \bar{R}_\xi) \mid \text{axl}(\bar{R}_\xi^T \partial_{\xi_2} \bar{R}_\xi) \mid \text{axl}(\bar{R}_\xi^T \partial_{\xi_3} \bar{R}_\xi) \right) \in \mathbb{R}^{3 \times 3}, \quad (3.10)$$

since (see [71]) the following close relationship between the wryness tensor and the dislocation density tensor holds

$$\alpha_\xi = -\Gamma_\xi^T + \text{tr}(\Gamma_\xi) \mathbb{1}_3, \quad \text{or equivalently,} \quad \Gamma_\xi = -\alpha_\xi^T + \frac{1}{2} \text{tr}(\alpha_\xi) \mathbb{1}_3. \quad (3.11)$$

For infinitesimal strains this formula is well-known under the name Nye's formula, and  $-\Gamma$  is also called Nye's curvature tensor. We will use this terminology further on [71].

Note that  $\partial_{x_k} \bar{Q}_e = \sum_{i=1}^3 \partial_{\xi_i} \bar{R}_\xi \partial_{x_k} \xi_i$ ,  $\partial_{\xi_k} \bar{R}_\xi = \sum_{i=1}^3 \partial_{x_i} \bar{Q}_e \partial_{\xi_k} x_i$  and

$$\begin{aligned} \bar{R}_\xi^T \partial_{\xi_k} \bar{R}_\xi &= \sum_{i=1}^3 (\bar{Q}_e^T \partial_{x_i} \bar{Q}_e) \partial_{\xi_k} x_i = \sum_{i=1}^3 (\bar{Q}_e^T \partial_{x_i} \bar{Q}_e) ([\nabla_x \Theta]^{-1})_{ik} \\ \text{axl}(\bar{R}_\xi^T \partial_{\xi_k} \bar{R}_\xi) &= \sum_{i=1}^3 \text{axl}(\bar{Q}_e^T \partial_{x_i} \bar{Q}_e) ([\nabla_x \Theta]^{-1})_{ik}. \end{aligned} \quad (3.12)$$

Thus, we have the chain rule

$$\begin{aligned} \Gamma_\xi &= \left( \sum_{i=1}^3 \text{axl}(\bar{Q}_e^T \partial_{x_i} \bar{Q}_e) ([\nabla_x \Theta]^{-1})_{i1} \mid \sum_{i=1}^3 \text{axl}(\bar{Q}_e^T \partial_{x_i} \bar{Q}_e) ([\nabla_x \Theta]^{-1})_{i2} \mid \sum_{i=1}^3 \text{axl}(\bar{Q}_e^T \partial_{x_i} \bar{Q}_e) ([\nabla_x \Theta]^{-1})_{i3} \right) \\ &= \left( \text{axl}(\bar{Q}_e^T \partial_{x_1} \bar{Q}_e) \mid \text{axl}(\bar{Q}_e^T \partial_{x_2} \bar{Q}_e) \mid \text{axl}(\bar{Q}_e^T \partial_{x_3} \bar{Q}_e) \right) [\nabla_x \Theta]^{-1}. \end{aligned} \quad (3.13)$$

Define  $\Gamma_e := \left( \text{axl}(\bar{Q}_e^T \partial_{x_1} \bar{Q}_e) \mid \text{axl}(\bar{Q}_e^T \partial_{x_2} \bar{Q}_e) \mid \text{axl}(\bar{Q}_e^T \partial_{x_3} \bar{Q}_e) \right)$ ,  $\alpha_e := \bar{Q}_e^T \text{Curl}_x \bar{Q}_e$ . Using Nye's formula for  $\alpha_e$  and  $\Gamma_e$ , we have the correspondence

$$\alpha_e = -\Gamma_e^T + \text{tr}(\Gamma_e) \mathbb{1}_3, \quad \text{or equivalently,} \quad \Gamma_e = -\alpha_e^T + \frac{1}{2} \text{tr}(\alpha_e) \mathbb{1}_3. \quad (3.14)$$

In view of (3.11) and (3.13), we can write

$$\begin{aligned} \alpha_\xi &= -\Gamma_\xi^T + \text{tr}(\Gamma_\xi) \mathbb{1}_3 = -(\Gamma_e [\nabla_x \Theta]^{-1})^T + \text{tr}(\Gamma_e [\nabla_x \Theta]^{-1}) \mathbb{1}_3 \\ &= -[\nabla_x \Theta]^{-T} \Gamma_e^T + \text{tr}(\Gamma_e) [\nabla_x \Theta]^{-T} - \text{tr}(\Gamma_e) [\nabla_x \Theta]^{-T} + \text{tr}(\Gamma_e [\nabla_x \Theta]^{-1}) \mathbb{1}_3 \\ &= [\nabla_x \Theta]^{-T} \alpha_e - \text{tr}(\Gamma_e) [\nabla_x \Theta]^{-T} + \text{tr}(\Gamma_e [\nabla_x \Theta]^{-1}) \mathbb{1}_3. \end{aligned} \quad (3.15)$$

Moreover

$$\begin{aligned} \text{tr}(\Gamma_e) &= -\text{tr}(\alpha_e) + \frac{3}{2} \text{tr}(\alpha_e) = \frac{1}{2} \text{tr}(\alpha_e), \\ \Gamma_e [\nabla_x \Theta]^{-1} &= -\alpha_e^T [\nabla_x \Theta]^{-1} + \frac{1}{2} \text{tr}(\alpha_e) [\nabla_x \Theta]^{-1}, \\ \text{tr}(\Gamma_e [\nabla_x \Theta]^{-1}) &= -\text{tr}(\alpha_e^T [\nabla_x \Theta]^{-1}) + \frac{1}{2} \text{tr}(\alpha_e) \text{tr}([\nabla_x \Theta]^{-1}). \end{aligned} \quad (3.16)$$

Thus, we deduce

$$\begin{aligned} \alpha_\xi &= [\nabla_x \Theta]^{-T} \alpha_e - \frac{1}{2} \text{tr}(\alpha_e) [\nabla_x \Theta]^{-T} - \text{tr}([\nabla_x \Theta]^{-T} \alpha_e) \mathbb{1}_3 + \frac{1}{2} \text{tr}(\alpha_e) \text{tr}([\nabla_x \Theta]^{-1}) \mathbb{1}_3 \\ &= [\nabla_x \Theta]^{-T} \alpha_e - \text{tr}(\alpha_e^T [\nabla_x \Theta]^{-1}) \mathbb{1}_3 - \frac{1}{2} \text{tr}(\alpha_e) \left\{ [\nabla_x \Theta]^{-T} - \text{tr}([\nabla_x \Theta]^{-1}) \mathbb{1}_3 \right\}. \end{aligned} \quad (3.17)$$

However, we will not use this formula to rewrite the curvature energy in the fictitious Cartesian configuration  $\Omega_h$ , since it is easier to use (from (3.11))

$$\begin{aligned} \text{sym} \alpha_\xi &= -\text{sym} \Gamma_\xi + \text{tr}(\Gamma_\xi) \mathbb{1}_3 = -\text{sym}(\Gamma_e [\nabla_x \Theta]^{-1}) + \text{tr}(\Gamma_e [\nabla_x \Theta]^{-1}) \mathbb{1}_3, \\ \text{dev sym} \alpha_\xi &= -\text{dev sym} \Gamma_\xi = -\text{dev sym}(\Gamma_e [\nabla_x \Theta]^{-1}), \\ \text{skew} \alpha_\xi &= -\text{skew} \Gamma_\xi = -\text{skew}(\Gamma_e [\nabla_x \Theta]^{-1}), \\ \text{tr}(\alpha_\xi) &= -\text{tr}(\Gamma_\xi) + 3 \text{tr}(\Gamma_\xi) = 2 \text{tr}(\Gamma_\xi) = 2 \text{tr}(\Gamma_e [\nabla_x \Theta]^{-1}), \end{aligned} \quad (3.18)$$

and to express the curvature energy in terms of  $\Gamma_e [\nabla_x \Theta]^{-1}$  as

$$W_{\text{curv}}(\alpha_\xi) = \mu L_c^2 \left( b_1 \|\text{dev sym}(\Gamma_e [\nabla_x \Theta]^{-1})\|^2 + b_2 \|\text{skew}(\Gamma_e [\nabla_x \Theta]^{-1})\|^2 + 4 b_3 [\text{tr}(\Gamma_e [\nabla_x \Theta]^{-1})]^2 \right). \quad (3.19)$$

Note that using

$$\bar{Q}_e^T \partial_{x_i} \bar{Q}_e = Q_0 \bar{R}^T \partial_{x_i} (\bar{R} Q_0^T) = Q_0 (\bar{R}^T \partial_{x_i} \bar{R}) Q_0^T - Q_0 (Q_0^T \partial_{x_i} Q_0) Q_0^T, \quad i = 1, 2, 3 \quad (3.20)$$

and the identity

$$\text{axl}(Q A Q^T) = Q \text{axl}(A) \quad \forall Q \in \text{SO}(3) \quad \text{and} \quad \forall A \in \mathfrak{so}(3), \quad (3.21)$$

we obtain the following form of the wryness tensor

$$\begin{aligned} \Gamma(x_1, x_2, x_3) &:= \Gamma_\xi(\Theta(x_1, x_2, x_3)) = \Gamma_e [\nabla_x \Theta]^{-1} \\ &= Q_0 \left[ \left( \text{axl}(\bar{R}^T \partial_{x_1} \bar{R}) \mid \text{axl}(\bar{R}^T \partial_{x_2} \bar{R}) \mid \text{axl}(\bar{R}^T \partial_{x_3} \bar{R}) \right) \right. \\ &\quad \left. - \left( \text{axl}(Q_0^T \partial_{x_1} Q_0) \mid \text{axl}(Q_0^T \partial_{x_2} Q_0) \mid \text{axl}(Q_0^T \partial_{x_3} Q_0) \right) \right] [\nabla_x \Theta]^{-1}. \end{aligned} \quad (3.22)$$

Applying the change of variables formula we obtain now a new form of the energy functional  $I$  which suggests to seek the unknown functions  $\varphi$  and  $\bar{R}$  as solutions of the following minimization problem

$$I = \int_{\Omega_h} \left[ \widetilde{W}_{\text{mp}}(F, \bar{R}) + \widetilde{W}_{\text{curv}}(\Gamma) \right] \det(\nabla_x \Theta) dV - \widetilde{\Pi}(\varphi, \bar{R}) \rightarrow \min. \quad \text{w.r.t.} \quad (\varphi, \bar{R}), \quad (3.23)$$

where  $dV$  denotes the volume element  $dx_1 dx_2 dx_3$  and

$$\begin{aligned} \widetilde{W}_{\text{mp}}(F, \bar{R}) &= W_{\text{mp}}(\bar{U}_e) = \mu \|\text{sym}(\bar{U}_e - \mathbb{1}_3)\|^2 + \mu_c \|\text{skew}(\bar{U}_e - \mathbb{1}_3)\|^2 + \frac{\lambda}{2} [\text{tr}(\text{sym}(\bar{U}_e - \mathbb{1}_3))]^2 \\ &= \mu \|\text{dev sym}(\bar{U}_e - \mathbb{1}_3)\|^2 + \mu_c \|\text{skew}(\bar{U}_e - \mathbb{1}_3)\|^2 + \frac{\kappa}{2} [\text{tr}(\text{sym}(\bar{U}_e - \mathbb{1}_3))]^2, \\ \widetilde{W}_{\text{curv}}(\Gamma) &= \mu L_c^2 (b_1 \|\text{dev sym} \Gamma\|^2 + b_2 \|\text{skew} \Gamma\|^2 + 4 b_3 [\text{tr}(\Gamma)]^2). \end{aligned}$$

The external loading potential can be written as

$$\widetilde{\Pi}(\varphi, \bar{R}) = \widetilde{\Pi}_f(\varphi) + \widetilde{\Pi}_t(\varphi) + \widetilde{\Pi}_{\Omega_h}(\bar{R}) + \widetilde{\Pi}_{\Gamma_t}(\bar{R}), \quad (3.24)$$

with

$$\begin{aligned} \widetilde{\Pi}_f(\varphi) &:= \Pi_f(\varphi_\xi) = \int_{\Omega_\xi} \langle f, u \rangle dV(\xi) = \int_{\Omega_h} \langle \tilde{f}, \tilde{u} \rangle dV, \\ \widetilde{\Pi}_t(\varphi) &:= \Pi_t(\varphi_\xi) = \int_{\partial\Omega_t} \langle t, u \rangle dS(\xi) = \int_{\Gamma_t} \langle \tilde{t}, \tilde{u} \rangle dS, \quad \widetilde{\Pi}_{\Omega_h}(\bar{R}) := \Pi_{\Omega}(\bar{R}_\xi), \quad \widetilde{\Pi}_{\Gamma_t}(\bar{R}) := \Pi_{\partial\Omega_t}(\bar{R}_\xi), \end{aligned} \quad (3.25)$$

where  $\tilde{u}(x_i) = \varphi(x_i) - \Theta(x_i)$  is the displacement vector and the vector fields  $\tilde{f}$  and  $\tilde{t}$  can be determined in terms of  $f$  and  $t$ , respectively, for instance  $(\tilde{f}(x))_i = (f(\Theta(x)))_i \det(\nabla_x \Theta)$ . Here,  $\Gamma_t$  and  $\Gamma_d$  are nonempty subsets of the boundary of  $\Omega_h$  such that  $\Gamma_t \cup \Gamma_d = \partial\Omega_h$  and  $\Gamma_t \cap \Gamma_d = \emptyset$ . On  $\Gamma_t$  we consider traction boundary conditions, while on  $\Gamma_d$  we have Dirichlet-type boundary conditions (i.e.,  $\varphi$  and  $\bar{R}$  are prescribed on  $\Gamma_d$ ). We assume that  $\Gamma_d$  has the form  $\Gamma_d = \gamma_d \times (-\frac{h}{2}, \frac{h}{2})$ , where the curve  $\gamma_d$  is a subset of  $\partial\omega$  with length  $(\gamma_d) > 0$ . Accordingly, the boundary subset  $\Gamma_t$  has the form  $\Gamma_t = \left( \gamma_t \times (-\frac{h}{2}, \frac{h}{2}) \right) \cup \left( \omega \times \left\{ \frac{h}{2} \right\} \right) \cup \left( \omega \times \left\{ -\frac{h}{2} \right\} \right)$  and  $\Theta(\Gamma_t) = \partial\Omega_t$ .

### 3.2 Useful tensors defined through the diffeomorphism $\Theta$

For our purpose, the diffeomorphism  $\Theta : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  describing the reference configuration (i.e. the curved surface of the shell), will be chosen in the specific form

$$\Theta(x_1, x_2, x_3) = y_0(x_1, x_2) + x_3 n_0(x_1, x_2), \quad n_0 = \frac{\partial_{x_1} y_0 \times \partial_{x_2} y_0}{\|\partial_{x_1} y_0 \times \partial_{x_2} y_0\|}, \quad (3.26)$$

where  $y_0 : \omega \rightarrow \mathbb{R}^3$  is a function of class  $C^2(\omega)$ . This specific form of the diffeomorphism  $\Theta$  maps the midsurface  $\omega$  of the fictitious Cartesian configuration parameter space  $\Omega_h$  onto the midsurface  $\omega_\xi = y_0(\omega)$  of  $\Omega_\xi$  and  $n_0$  is the unit normal vector to  $\omega_\xi$ . For simplicity and where no confusions may arise, further on we will omit to write explicitly the arguments  $(x_1, x_2, x_3)$  of the diffeomorphism  $\Theta$  or we will specify only its dependence on  $x_3$ .

Remark that

$$\nabla_x \Theta(x_3) = (\nabla y_0 | n_0) + x_3 (\nabla n_0 | 0) \quad \forall x_3 \in \left( -\frac{h}{2}, \frac{h}{2} \right), \quad \nabla_x \Theta(0) = (\nabla y_0 | n_0), \quad [\nabla_x \Theta(0)]^{-T} \cdot e_3 = n_0, \quad (3.27)$$

and  $\det(\nabla y_0 | n_0) = \sqrt{\det[(\nabla y_0)^T \nabla y_0]}$  represents the surface element.

In the following we identify the *Weingarten map* (or *shape operator*) on  $y_0(\omega)$  with its associated matrix  $L_{y_0} \in \mathbb{R}^{2 \times 2}$  defined in Appendix A.2 by  $L_{y_0} = I_{y_0}^{-1} \Pi_{y_0}$ , where  $I_{y_0}$  and  $\Pi_{y_0}$  are the matrix representations of the *first fundamental form* (metric) and the *second fundamental form* on  $y_0(\omega)$ , respectively. Then, the *Gauß curvature*  $K$  of the surface  $y_0(\omega)$  is determined by  $K = \det(L_{y_0})$  and the *mean curvature*  $H$  through  $2H := \text{tr}(L_{y_0})$ . We denote the principal curvatures of the surface by  $\kappa_1$  and  $\kappa_2$ .

For our purpose we will write the expressions of  $\nabla_x \Theta$ ,  $\det(\nabla_x \Theta)$ ,  $[\nabla_x \Theta]^{-1}$  corresponding to the special form of the map  $\Theta$  given by (3.26), as well as some other of its properties, see Appendix A.3. We have  $[\nabla_x \Theta(x_3)] \cdot e_3 = n_0$ . Let us recall that  $X \in \text{GL}^+(3)$  satisfies the *Generalized Kirchhoff Constraint* (GKC) [59] if  $X \in \text{GKC} := \{X \in \text{GL}^+(3) \mid X^T X \cdot e_3 = \varrho^2 e_3, \varrho \in \mathbb{R}^+\}$ . For all  $X \in \text{GKC}$  with the polar decomposition  $X = R U_0$ , it follows that  $U_0 \in \text{GKC}$ . In view of this property and  $\nabla \Theta(x_3) = Q_0(x_3) U_0(x_3)$ , it follows<sup>2</sup>

$$U_0(x_3) = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{which implies that}$$

$$d_3^0(x_3) := Q_0(x_3) \cdot e_3 = Q_0(x_3) U_0(x_3) \cdot e_3 = \nabla_x \Theta(x_3) \cdot e_3 = n_0. \quad (3.28)$$

This means that the initial director  $d_3^0$  is chosen along the normal to the reference midsurface (the “*material filament*” of the shell, see Figure 2), while  $d_\alpha^0 := Q_0 \cdot e_\alpha$ , for  $\alpha = 1, 2$ , is an orthonormal basis in the tangent plane of  $\omega_\xi$ . In the current configuration  $\Omega_c$  the director  $d_3 := \overline{Q}_e \cdot d_3^0$  is no longer orthogonal to the deformed surface  $\omega_c := \varphi_\xi(\omega_\xi)$  and the directors  $d_\alpha := \overline{Q}_e \cdot d_\alpha^0$  are not tangent to this surface. The deviation of the director  $d_3$  from the normal vector to  $\omega_c$  describes the *transverse shear deformation* of shells. Moreover, the rotations of  $d_1, d_2$  about the director  $d_3$  describe the so-called *drilling rotations* in shells (see [8, 97]).

Let us introduce the tensors<sup>3</sup> defined by:

$$A_{y_0} := (\nabla y_0 | 0) [\nabla_x \Theta(0)]^{-1} \in \mathbb{R}^{3 \times 3}, \quad B_{y_0} := -(\nabla n_0 | 0) [\nabla_x \Theta(0)]^{-1} \in \mathbb{R}^{3 \times 3}, \quad (3.29)$$

and the so-called *alternator tensor*  $C_{y_0}$  of the surface [99]

$$C_{y_0} := \det(\nabla_x \Theta(0)) [\nabla_x \Theta(0)]^{-T} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} [\nabla_x \Theta(0)]^{-1}. \quad (3.30)$$

The introduced tensors have the properties given by Proposition A.3 from Appendix A.4, which are essential in the derivation of the model entirely in matrix representation.

### 3.3 Plane stress conditions in the curved (reference) configuration

The *first Piola-Kirchhoff stress tensor* in the reference (curved) configuration  $\Omega_\xi$  is given by  $S_1(F_\xi, \overline{R}_\xi) = D_{F_\xi} \widetilde{W}_{\text{mp}}(F_\xi, \overline{R}_\xi)$ . We also consider the *Biot-type stress tensor* from classical elasticity theory  $T_{\text{Biot}}(\overline{U}_\xi) := D_{\overline{U}_\xi} W_{\text{mp}}(\overline{U}_\xi)$ . Since  $D_{F_\xi} \overline{U}_\xi \cdot X = \overline{R}_\xi^T X$  and  $\langle D_{F_\xi} \widetilde{W}_{\text{mp}}(F_\xi, \overline{R}_\xi), X \rangle = \langle D_{\overline{U}_\xi} W_{\text{mp}}(\overline{U}_\xi), D_{F_\xi} \overline{U}_\xi \cdot X \rangle$ , for all  $X \in \mathbb{R}^{3 \times 3}$ , we note that  $D_{F_\xi} \widetilde{W}_{\text{mp}}(F_\xi, \overline{R}_\xi) = \overline{R}_\xi D_{\overline{U}_\xi} W_{\text{mp}}(\overline{U}_\xi)$ . Hence, we have

$$S_1(F_\xi, \overline{R}_\xi) = \overline{R}_\xi T_{\text{Biot}}(\overline{U}_\xi), \quad T_{\text{Biot}}(\overline{U}_\xi) = \overline{R}_\xi^T S_1(F_\xi, \overline{R}_\xi). \quad (3.31)$$

The Biot-type stress tensor  $T_{\text{Biot}}$  is given by

$$T_{\text{Biot}}(\overline{U}_\xi) = 2\mu \text{sym}(\overline{U}_\xi - \mathbb{1}_3) + 2\mu_c \text{skew}(\overline{U}_\xi - \mathbb{1}_3) + \lambda \text{tr}(\text{sym}(\overline{U}_\xi - \mathbb{1}_3)) \mathbb{1}_3, \quad (3.32)$$

<sup>2</sup>In the rest of the paper \* denotes quantities having expressions which are not relevant for our calculations.

<sup>3</sup>These tensors are usually called the first fundamental form and the second fundamental form, respectively. However, we will not use this terminology since it may lead to some confusions. The relation between these tensors are explained in Proposition A.3.

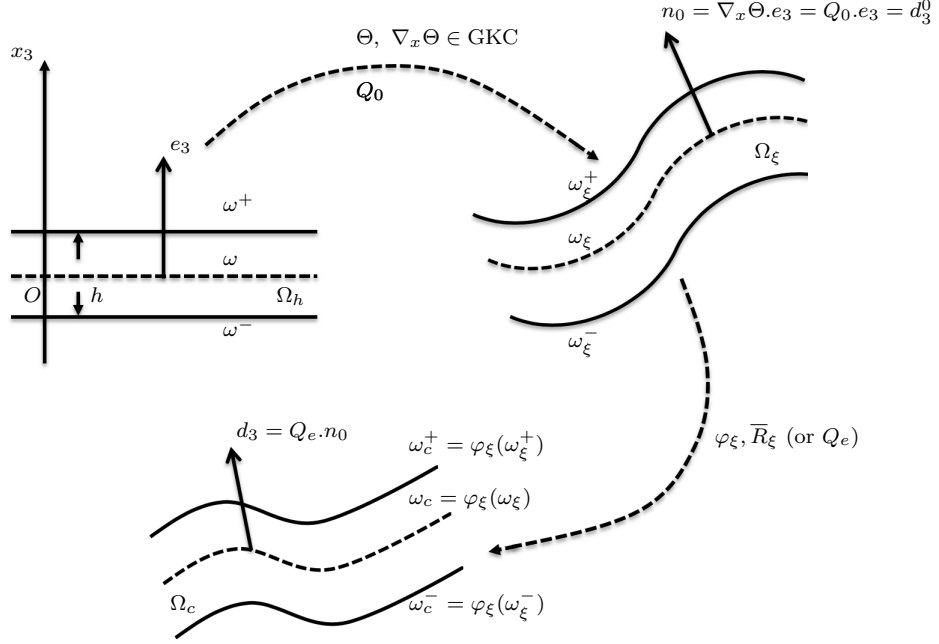


Figure 2: Transverse section in the shell. The shell is stress free at the upper and lower surface in the current configuration  $\Omega_c$ . With regard to the first Piola-Kirchhoff tensor  $S_1$  this is equivalent to condition (3.34).

while, using (3.31), we obtain that the first Piola-Kirchhoff tensor  $S_1$  has the following form

$$S_1(F_\xi, \bar{R}_\xi) = \bar{R}_\xi \left[ 2 \mu \text{sym}(\bar{R}_\xi^T F_\xi - \mathbb{1}_3) + 2 \mu_c \text{skew}(\bar{R}_\xi^T F_\xi - \mathbb{1}_3) + \lambda \text{tr}(\text{sym}(\bar{R}_\xi^T F_\xi - \mathbb{1}_3)) \mathbb{1}_3 \right]. \quad (3.33)$$

As usual in the development of shell theories, we assume that the normal stress (Piola-Kirchhoff stress tensor in the normal direction  $n_0$ ) on the transverse boundaries (*upper and lower faces*  $\omega_\xi^+$  and  $\omega_\xi^-$ , respectively, of the curved reference configuration  $\Omega_\xi$ ) are vanishing, i.e.

$$S_1(F_\xi, \bar{R}_\xi) \Big|_{\omega_\xi^\pm} \cdot (\pm n_0) = 0 \quad \text{“zero normal stresses” on upper and lower faces.} \quad (3.34)$$

In the limit case  $h \rightarrow \infty$ , these conditions imply  $S_1(F_\xi, \bar{R}_\xi) \Big|_{\omega_\xi} \cdot (\pm n_0) = 0$ , i.e. “zero normal stresses” on the midsurface  $\omega_\xi = \Theta(\omega \times \{0\})$ , but the reverse of this implication is not valid.

In fact, (3.34) is equivalent to the assumption that the Biot-stress tensor in the normal direction  $n_0$  is vanishing, since  $T_{\text{Biot}}(\bar{U}_\xi) \Big|_{\omega_\xi^\pm} \cdot (\pm n_0) = \bar{R}_\xi^T \Big|_{\omega_\xi^\pm} S_1(F_\xi, \bar{R}_\xi) \Big|_{\omega_\xi^\pm} \cdot (\pm n_0) = 0$ , and this implies, after scalar multiplication with  $n_0$

$$\langle T_{\text{Biot}}(\bar{U}_\xi) \Big|_{\omega_\xi^\pm} \cdot n_0, n_0 \rangle = 0 \quad \text{“zero normal tractions” on upper and lower faces.} \quad (3.35)$$

### 3.4 Neumann boundary conditions in the fictitious Cartesian configuration

Using the coordinates of the fictitious Cartesian configuration, the plane stress conditions (3.35) are written in the form

$$\langle T_{\text{Biot}} \left( \bar{U}_e(x_1, x_2, \pm \frac{h}{2}) \right) \cdot n_0, n_0 \rangle = 0. \quad (3.36)$$

A simplified approximated form of (3.36) can be written in the limit case  $h \rightarrow 0$  as in the following. Let us define the function

$$f(x_3) := \langle T_{\text{Biot}}(\bar{U}_e(x_1, x_2, x_3)) \cdot n_0, n_0 \rangle, \quad \forall x_3 \in \left[ -\frac{h}{2}, \frac{h}{2} \right]. \quad (3.37)$$

The Taylor expansion of  $f(x_3)$  in  $x_3 = 0$  leads to

$$f\left(\frac{h}{2}\right) + f\left(-\frac{h}{2}\right) = 2f(0) + O(h^2), \quad f\left(\frac{h}{2}\right) - f\left(-\frac{h}{2}\right) = hf'(0) + O(h^3), \quad (3.38)$$

where

$$f'(0) = \langle \partial_{x_3} T_{\text{Biot}}(\bar{U}_e(x_1, x_2, x_3)) \Big|_{x_3=0} \cdot n_0, n_0 \rangle. \quad (3.39)$$

In view of the boundary conditions (3.36) we have  $f\left(\frac{h}{2}\right) = 0 = f\left(-\frac{h}{2}\right)$  and the relations (3.38) yield  $f(0) = O(h^2)$  and  $f'(0) = O(h^2)$ . In the limits case  $h \rightarrow 0$  one obtains the following approximated form of the conditions (3.36)

$$\begin{aligned} \langle T_{\text{Biot}}(\bar{U}_e(x_1, x_2, 0)) \cdot n_0, n_0 \rangle &= 0 \quad \text{“zero normal tractions” on the midsurface } \omega_\xi = \Theta(\omega), \quad (3.40) \\ \langle \partial_{x_3} T_{\text{Biot}}(\bar{U}_e(x_1, x_2, x_3)) \Big|_{x_3=0} \cdot n_0, n_0 \rangle &= 0 \quad \text{“zero variations of normal tractions” on the midsurface } \omega_\xi = \Theta(\omega). \end{aligned}$$

These relations represent a first approximation in the dimensional reduction procedure and they will be used further on. In addition, in Appendix A.5 we prove that, for our method, this first approximation leads to the same results (but in a simpler way) as in the case when the complete Neumann condition (3.34) are used, and then the limit  $h \rightarrow 0$  is considered.

## 4 The two-dimensional approximation

### 4.1 The 8-parameter ansatz for the two-dimensional approximation

In the following, we want to find a *reasonable approximation* of  $(\varphi, \bar{R})$  involving only two-dimensional quantities. Following the formal dimensional reduction procedure for the Cosserat elastic plates given in [58], we consider that the rotation  $\bar{R} : \Omega_h \rightarrow \text{SO}(3)$  in the thin shell does not depend on the thickness variable  $x_3$

$$\bar{R}(x_1, x_2, x_3) = \bar{R}_s(x_1, x_2), \quad (4.1)$$

in line with the assumed thinness and material homogeneity of the structure. Moreover, an approximation of the elastic rotation  $\bar{Q}_e : \Omega_h \rightarrow \text{SO}(3)$  will be given by  $\bar{Q}_{e,s}$

$$\bar{Q}_{e,s}(x_1, x_2) = \bar{R}_s(x_1, x_2) Q_0^T(x_1, x_2, 0). \quad (4.2)$$

Taking into account<sup>4</sup> that  $\nabla_x \Theta \in \text{GKC}$ , with  $\varrho = 1$ , we have  $[\nabla_x \Theta]^{-1} [\nabla_x \Theta]^{-T} \cdot e_3 = e_3$ . In view of the properties of GKC, it follows

$$\begin{aligned} \bar{R}_s(x_1, x_2) \cdot e_3 &= \bar{R}_s(x_1, x_2) U_0(x_1, x_2, 0) \cdot e_3 = \bar{Q}_{e,s}(x_1, x_2) Q_0(x_1, x_2, 0) U_0(x_1, x_2, 0) \cdot e_3 \\ &= \bar{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0) \cdot e_3 = \bar{Q}_{e,s}(x_1, x_2) (\nabla y_0 | n_0) \cdot e_3 = \bar{Q}_{e,s}(x_1, x_2) \cdot n_0. \end{aligned} \quad (4.3)$$

Since  $\bar{Q}_{e,s} Q_0 \in \text{SO}(3)$  and with (3.28) we have

$$Q_0(x_1, x_2, x_3) \cdot e_3 = n_0 = Q_0(x_1, x_2, 0) \cdot e_3. \quad (4.4)$$

In the engineering shell community it is well known [18, 85, 74] that the ansatz for the deformation over the thickness should be at least quadratic in order to avoid the so called *Poisson thickness locking* and to fully capture the three-dimensional kinematics without artificial modification of the material laws<sup>5</sup>, see the detailed discussion of this point in [10] and compare with [14, 13, 80, 9, 83].

<sup>4</sup>The definition of the set GKC and its properties are presented in Appendix A.4

<sup>5</sup>Let us quote from [85]: “Due to bending this change of length is generally asymmetric about (the midsurface) and leads to a shift of the original midsurfaces.... This asymmetry requires at least a quadratic representation of the (deformation in thickness direction).”

We consider therefore the following *8-parameter quadratic ansatz* in the thickness direction for the reconstructed total deformation  $\varphi_s : \Omega_h \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$  of the shell-like structure

$$\varphi_s(x_1, x_2, x_3) = m(x_1, x_2) + \left( x_3 \varrho_m(x_1, x_2) + \frac{x_3^2}{2} \varrho_b(x_1, x_2) \right) \overline{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0) \cdot e_3, \quad (4.5)$$

where  $m : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$  takes on the role of the deformation of the midsurface of the shell viewed as a parametrized surface, the yet indeterminate functions  $\varrho_m, \varrho_b : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$  allow in principal for symmetric thickness stretch ( $\varrho_m \neq 1$ ) and asymmetric thickness stretch ( $\varrho_b \neq 0$ ) about the midsurface.

To construct the new geometrically nonlinear Cosserat shell model we start with an 8-parameter ansatz (8 ‘dof’: 3 components of the membrane deformation  $m$ , 3 degrees of freedom for  $\overline{R} \in \text{SO}(3)$ , 2 degrees of freedom over the thickness  $\varrho_m$  and  $\varrho_b$ ) but in the end we will arrive at a *6-parameter model*. This will be possible because in the isotropic case the two scalar parameters  $\varrho_m$  and  $\varrho_b$  (the degrees of freedom over the thickness) can analytically be condensed out.

In view of (4.4), the above 8-parameter quadratic ansatz in the thickness direction can be written as

$$\varphi_s(x_1, x_2, x_3) = m(x_1, x_2) + \left( x_3 \varrho_m(x_1, x_2) + \frac{x_3^2}{2} \varrho_b(x_1, x_2) \right) \overline{Q}_{e,s}(x_1, x_2) \cdot n_0. \quad (4.6)$$

With regard to the total deformation, this is then a kind of plate formulation since the midsurface of the fictitious Cartesian reference configuration  $\omega \subset \mathbb{R}^2$  is assumed to lie in the plane. This implies for the total (reconstructed) deformation gradient of the shell

$$\begin{aligned} F_s = \nabla_x \varphi_s(x_1, x_2, x_3) &= (\nabla m | \varrho_m \overline{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0) \cdot e_3) \\ &+ x_3 (\nabla [\varrho_m \overline{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0) \cdot e_3] | \varrho_b \overline{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0) \cdot e_3) \\ &+ \frac{x_3^2}{2} (\nabla [\varrho_b \overline{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0) \cdot e_3] | 0) \end{aligned} \quad (4.7)$$

and especially

$$\begin{aligned} F_s \cdot e_3 &= \varrho_m \overline{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0) \cdot e_3 + x_3 \varrho_b \overline{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0) \cdot e_3 \\ &\stackrel{(4.3)}{=} (\varrho_m + x_3 \varrho_b) \overline{R}_s(x_1, x_2) \cdot e_3. \end{aligned} \quad (4.8)$$

If  $\overline{Q}_e = \mathbb{1}_3$ ,  $\varrho_m = 1$ ,  $\varrho_b = 0$ ,  $m = y_0$  (as in the reference configuration  $\Omega_\xi$ ), then  $F_s = \nabla_x \Theta$ .

In the rest of the paper we do not write explicitly the dependence of these functions on  $x_1$  and  $x_2$ .

## 4.2 From an 8-parameter ansatz to a 6-parameter model via the fictitious boundary conditions

We intend to find  $\varrho_m$  and  $\varrho_b$  such that the boundary conditions (3.40) in the fictitious configuration

$$\langle T_{\text{Biot}}(\overline{U}_{e,s}) \cdot n_0, n_0 \rangle = 0, \quad \langle \partial_{x_3} T_{\text{Biot}}(\overline{U}_{e,s}) \cdot n_0, n_0 \rangle = 0 \quad \text{for} \quad x_3 = 0$$

are satisfied. These boundary conditions are equivalent to

$$\langle T_{\text{Biot}}(\overline{U}_{e,s}) Q_0 \cdot e_3, Q_0 \cdot e_3 \rangle = 0, \quad \langle \partial_{x_3} T_{\text{Biot}}(\overline{U}_{e,s}) Q_0 \cdot e_3, Q_0 \cdot e_3 \rangle = 0 \quad \text{for} \quad x_3 = 0, \quad (4.9)$$

and further to

$$\langle Q_0^T T_{\text{Biot}}(\overline{U}_{e,s}) Q_0 \cdot e_3, e_3 \rangle = 0, \quad \langle Q_0^T \partial_{x_3} T_{\text{Biot}}(\overline{U}_{e,s}) Q_0 \cdot e_3, e_3 \rangle = 0 \quad \text{for} \quad x_3 = 0, \quad (4.10)$$

where

$$\begin{aligned} \overline{U}_{e,s} &= \overline{Q}_{e,s}^T F_s [\nabla_x \Theta]^{-1} = Q_0 \overline{R}^T F_s [\nabla_x \Theta]^{-1}, \\ T_{\text{Biot}}(\overline{U}_{e,s}) &= 2 \mu \text{sym}(\overline{U}_{e,s} - \mathbb{1}_3) + 2 \mu_c \text{skew}(\overline{U}_{e,s} - \mathbb{1}_3) + \lambda \text{tr}(\text{sym}(\overline{U}_{e,s} - \mathbb{1}_3)) \mathbb{1}_3. \end{aligned} \quad (4.11)$$



To this aim, we calculate

$$\begin{aligned}
2 Q_0^T \text{sym}(\bar{U}_{e,s} - \mathbb{1}_3) Q_0 &= Q_0^T \left( \bar{Q}_{e,s}^T F_s [\nabla_x \Theta]^{-1} + [\nabla_x \Theta]^{-T} F_s^T \bar{Q}_{e,s} - 2 \mathbb{1}_3 \right) Q_0 \\
&= Q_0^T \bar{Q}_{e,s}^T F_s U_0^{-1} Q_0^T Q_0 + Q_0^T Q_0 U_0^{-T} F_s^T \bar{Q}_{e,s} Q_0 - 2 \mathbb{1}_3 \\
&= Q_0^T \bar{Q}_{e,s}^T F_s U_0^{-1} + U_0^{-T} F_s^T \bar{Q}_{e,s} Q_0 - 2 \mathbb{1}_3,
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
2 Q_0^T \text{skew}(\bar{U}_{e,s} - \mathbb{1}_3) Q_0 &= Q_0^T \left( \bar{Q}_{e,s}^T F_s [\nabla_x \Theta]^{-1} - [\nabla_x \Theta]^{-T} F_s^T \bar{Q}_{e,s} \right) Q_0 \\
&= Q_0^T \bar{Q}_{e,s}^T F_s U_0^{-1} Q_0^T Q_0 - Q_0^T Q_0 U_0^{-T} F_s^T \bar{Q}_{e,s} Q_0 = Q_0^T \bar{Q}_{e,s}^T F_s U_0^{-1} - U_0^{-T} F_s^T \bar{Q}_{e,s} Q_0,
\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}
\langle Q_0^T \text{sym}(\bar{U}_{e,s} - \mathbb{1}_3) Q_0, e_3, e_3 \rangle &= \langle (Q_0^T \bar{Q}_{e,s}^T F_s U_0^{-1} + U_0^{-T} F_s^T \bar{Q}_{e,s} Q_0 - 2 \mathbb{1}_3), e_3, e_3 \rangle \\
&= 2 \langle (U_0^{-1} F_s^T \bar{Q}_{e,s} Q_0 - \mathbb{1}_3), e_3, e_3 \rangle = 2 \langle U_0^{-1} F_s^T \bar{Q}_{e,s} Q_0, e_3, e_3 \rangle - 2, \\
&= 2 \langle F_s^T \bar{Q}_{e,s} Q_0, e_3, U_0^{-T} e_3 \rangle - 2 = 2 \langle F_s^T \bar{Q}_{e,s} Q_0, e_3, e_3 \rangle - 2 \\
&\stackrel{(4.3)}{=} 2 \langle F_s^T \bar{R}_s, e_3, e_3 \rangle - 2 = 2 \langle \bar{R}_s, e_3, F_s, e_3 \rangle - 2 \\
&\stackrel{(4.8)}{=} 2 \langle \bar{R}_s, e_3, (\varrho_m + x_3 \varrho_b) \bar{R}_s, e_3 \rangle - 2 = 2 (\varrho_m + x_3 \varrho_b - 1)
\end{aligned} \tag{4.14}$$

where we have used the special structure of  $U_0 = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}$  and  $\bar{R}_s \in \text{SO}(3)$ . Furthermore, we have

$$\langle Q_0^T \text{skew}(\bar{U}_{e,s} - \mathbb{1}_3) Q_0, e_3, e_3 \rangle = \langle (Q_0^T \bar{Q}_{e,s}^T F_s U_0^{-1} - U_0^{-T} F_s^T \bar{Q}_{e,s} Q_0), e_3, e_3 \rangle = 0, \tag{4.15}$$

and

$$\begin{aligned}
\langle Q_0^T \text{tr}(\text{sym}(\bar{U}_{e,s} - \mathbb{1}_3)) \mathbb{1}_3 Q_0, e_3, e_3 \rangle &= \text{tr}(\text{sym}(\bar{U}_{e,s} - \mathbb{1}_3)) \langle Q_0^T Q_0, e_3, e_3 \rangle = \text{tr}(\text{sym}(\bar{U}_{e,s} - \mathbb{1}_3)) \\
&= \langle \text{sym}(\bar{U}_{e,s} - \mathbb{1}_3), \mathbb{1}_3 \rangle = \langle Q_0^T \bar{Q}_{e,s}^T F_s U_0^{-1} + U_0^{-T} F_s^T \bar{Q}_{e,s} Q_0 - 2 \mathbb{1}_3, \mathbb{1}_3 \rangle \\
&= 2 \langle U_0^{-1} F_s^T \bar{Q}_{e,s} Q_0 - \mathbb{1}_3, \mathbb{1}_3 \rangle = 2 \langle F_s^T \bar{Q}_{e,s} Q_0, U_0^{-1} \rangle - 6 \\
&= 2 \left[ \langle (\nabla m | 0)^T \bar{Q}_{e,s} Q_0(x_3), U_0^{-1} \rangle + \varrho_m + x_3 \varrho_m \langle (\nabla(\bar{Q}_{e,s} Q_0(x_3) \cdot e_3) | 0)^T \bar{Q}_{e,s} Q_0(x_3), U_0^{-1} \rangle \right. \\
&\quad \left. + x_3 \varrho_b + \frac{x_3^2}{2} \varrho_b \langle (\nabla(\bar{Q}_{e,s} Q_0(x_3) \cdot e_3) | 0)^T \bar{Q}_{e,s} Q_0(x_3), U_0^{-1} \rangle - 3 \right] \\
&= 2 \left[ \langle (\nabla m | 0)^T \bar{Q}_{e,s}, [\nabla_x \Theta(x_3)]^{-1} \rangle + \varrho_m + x_3 \varrho_m \langle (\nabla(\bar{Q}_{e,s} Q_0(x_3) \cdot e_3) | 0)^T \bar{Q}_{e,s}, [\nabla_x \Theta(x_3)]^{-1} \rangle \right. \\
&\quad \left. + x_3 \varrho_b + \frac{x_3^2}{2} \varrho_b \langle (\nabla(\bar{Q}_{e,s} Q_0(x_3) \cdot e_3) | 0)^T \bar{Q}_{e,s}, [\nabla_x \Theta(x_3)]^{-1} \rangle - 3 \right].
\end{aligned} \tag{4.16}$$

We deduce

$$\begin{aligned}
\langle Q_0^T T_{\text{Biot}}(\bar{U}_{e,s}) Q_0, e_3, e_3 \rangle &= \mu [2(\varrho_m - 1) + 2x_3 \varrho_b] + \lambda \left[ \langle (\nabla m | 0)^T \bar{Q}_{e,s}, [\nabla_x \Theta(x_3)]^{-1} \rangle + \varrho_m \right. \\
&\quad \left. + x_3 \varrho_m \langle (\nabla(\bar{Q}_{e,s} Q_0(x_3) \cdot e_3) | 0)^T \bar{Q}_{e,s}, [\nabla_x \Theta(x_3)]^{-1} \rangle \right. \\
&\quad \left. + x_3 \varrho_b + \frac{x_3^2}{2} \varrho_b \langle \bar{Q}_{e,s}^T (\nabla(\bar{Q}_{e,s} Q_0(x_3) \cdot e_3) | 0), [\nabla_x \Theta(x_3)]^{-T} \rangle - 3 \right].
\end{aligned} \tag{4.17}$$

The requirement (4.10)<sub>1</sub> leads to the ‘‘plane stress’’ requirement for  $x_3 = 0$  (zero normal tractions on the upper and lower surface)

$$2\mu(\varrho_m - 1) + \lambda \left[ \langle (\nabla m | 0)^T \bar{Q}_{e,s}, [\nabla_x \Theta(0)]^{-1} \rangle + \varrho_m - 3 \right] = 0, \tag{4.18}$$

which, considering  $\nabla_x \Theta = (\nabla y_0 | n_0) + x_3 (\nabla n_0 | 0)$ , is equivalent to

$$2\mu(\varrho_m - 1) + \lambda \left[ \langle \bar{Q}_{e,s}^T (\nabla m | 0), (\nabla y_0 | n_0)^{-T} \rangle + \varrho_m - 3 \right] = 0 \tag{4.19}$$

and we obtain

$$\varrho_m = 1 - \frac{\lambda}{\lambda + 2\mu} [\langle \bar{Q}_{e,s}^T (\nabla m|0) [\nabla_x \Theta(0)]^{-1}, \mathbf{1}_3 \rangle - 2]. \quad (4.20)$$

Now, let us consider the boundary conditions (4.10)<sub>2</sub> and observe

$$\langle Q_0^T \partial_{x_3} T_{\text{Biot}}(\bar{U}_{e,s}) Q_0 \cdot e_3, e_3 \rangle = \langle \partial_{x_3} T_{\text{Biot}}(\bar{U}_{e,s}) Q_0 \cdot e_3, Q_0 \cdot e_3 \rangle = \partial_{x_3} \langle Q_0^T T_{\text{Biot}}(\bar{U}_{e,s}) Q_0 \cdot e_3, e_3 \rangle, \quad (4.21)$$

since  $Q_0 \cdot e_3 = n_0$  and therefore it is independent of  $x_3$ . We deduce

$$\begin{aligned} \partial_{x_3} \langle Q_0^T T_{\text{Biot}}(\bar{U}_{e,s}) Q_0 \cdot e_3, e_3 \rangle &= 2\mu \varrho_b + \lambda [\langle \bar{Q}_{e,s}^T (\nabla m|0), \partial_{x_3} [\nabla_x \Theta(x_3)]^{-T} \rangle \\ &\quad + \varrho_m \langle \bar{Q}_{e,s}^T (\nabla (\bar{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3)|0), [\nabla_x \Theta(x_3)]^{-T} \rangle \\ &\quad + x_3 \varrho_m \langle \bar{Q}_{e,s}^T (\nabla (\bar{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3)|0), \partial_{x_3} [\nabla_x \Theta(x_3)]^{-T} \rangle \\ &\quad + \varrho_b + x_3 \varrho_b \langle \bar{Q}_{e,s}^T (\nabla (\bar{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3)|0), [\nabla_x \Theta(x_3)]^{-T} \rangle - 3] \\ &\quad + \frac{x_3^2}{2} \varrho_b \langle \bar{Q}_{e,s}^T (\nabla (\bar{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3)|0), \partial_{x_3} [\nabla_x \Theta(x_3)]^{-T} \rangle \end{aligned} \quad (4.22)$$

and the boundary condition (4.21)<sub>2</sub> becomes

$$2\mu \varrho_b + \lambda [\langle \bar{Q}_{e,s}^T (\nabla m|0), \partial_{x_3} [\nabla_x \Theta(x_3)]^{-T} \rangle]_{x_3=0} + \varrho_m \langle \bar{Q}_{e,s}^T (\nabla (\bar{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3)|0), [\nabla_x \Theta(0)]^{-T} \rangle + \varrho_b = 0. \quad (4.23)$$

Here we have used that  $\partial_{x_3} [\nabla_x \Theta]^{-1} \Big|_{x_3=0}$  is finite, since  $\det(\nabla_x \Theta(x_3))$  has a third order polynomial expression, see Proposition A.2, and  $\det(\nabla y_0|n_0) \neq 0$ . We also remark that

$$\partial_{x_3} [\nabla_x \Theta(x_3)]^{-1} \Big|_{x_3=0} = -[\nabla_x \Theta(0)]^{-1} \partial_{x_3} [\nabla_x \Theta(x_3)] \Big|_{x_3=0} [\nabla_x \Theta(0)]^{-1} = -(\nabla y_0|n_0)^{-1} (\nabla n_0|0) (\nabla y_0|n_0)^{-1}.$$

Therefore, equation (4.21)<sub>2</sub> is equivalent to

$$\begin{aligned} 2\mu \varrho_b + \lambda [-\langle \bar{Q}_{e,s}^T (\nabla m|0) [\nabla_x \Theta(0)]^{-1} (\nabla n_0|0) [\nabla_x \Theta(0)]^{-1}, \mathbf{1}_3 \rangle \\ + \varrho_m \langle \bar{Q}_{e,s}^T (\nabla (\bar{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3)|0), [\nabla_x \Theta(0)]^{-T} \rangle + \varrho_b] = 0, \end{aligned} \quad (4.24)$$

which yields

$$\begin{aligned} \varrho_b &= \frac{\lambda}{\lambda + 2\mu} \langle \bar{Q}_{e,s}^T (\nabla m|0) [\nabla_x \Theta(0)]^{-1} (\nabla n_0|0) [\nabla_x \Theta(0)]^{-1}, \mathbf{1}_3 \rangle \\ &\quad - \varrho_m \frac{\lambda}{\lambda + 2\mu} \langle \bar{Q}_{e,s}^T (\nabla (\bar{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3)|0) [\nabla_x \Theta(0)]^{-1}, \mathbf{1}_3 \rangle \\ &= \frac{\lambda}{\lambda + 2\mu} \langle \bar{Q}_{e,s}^T (\nabla m|0) [\nabla_x \Theta(0)]^{-1} (\nabla n_0|0) [\nabla_x \Theta(0)]^{-1}, \mathbf{1}_3 \rangle \\ &\quad - \frac{\lambda}{\lambda + 2\mu} \langle \bar{Q}_{e,s}^T (\nabla (\bar{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3)|0) [\nabla_x \Theta(0)]^{-1}, \mathbf{1}_3 \rangle \\ &\quad + \frac{\lambda^2}{(\lambda + 2\mu)^2} \left[ \langle \bar{Q}_{e,s}^T (\nabla (\bar{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3)|0) [\nabla_x \Theta(0)]^{-1}, \mathbf{1}_3 \rangle \right] \left[ \langle \bar{Q}_{e,s}^T (\nabla m|0) [\nabla_x \Theta(0)]^{-1}, \mathbf{1}_3 \rangle - 2 \right]. \end{aligned} \quad (4.25)$$

**Remark 4.1.** The term  $\frac{\lambda^2}{(\lambda+2\mu)^2} \langle \bar{Q}_{e,s}^T (\nabla (\bar{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3)|0) [\nabla_x \Theta(0)]^{-1}, \mathbf{1}_3 \rangle [\langle \bar{Q}_{e,s}^T (\nabla m|0) [\nabla_x \Theta(0)]^{-1}, \mathbf{1}_3 \rangle - 2]$  represents a nonlinear coupling between midsurface in-plane (membrane) strain and normal curvature, a result of the derivation not present in the underlying three-dimensional theory where only products of deformation gradient and rotations occur<sup>6</sup>. Since we have in mind a small strain situation, this product is one order smaller than  $\frac{\lambda}{\lambda+2\mu} \langle (\nabla (\bar{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3)|0)^T \bar{Q}_{e,s} [\nabla_x \Theta(0)]^{-1}, \mathbf{1}_3 \rangle$ . Therefore, we neglect this term. The presence of the term  $-\frac{\lambda}{\lambda+2\mu} \langle \bar{Q}_{e,s}^T (\nabla m|0) (\nabla y_0|n_0)^{-1} (\nabla n_0|0) (\nabla y_0|n_0)^{-1}, \mathbf{1}_3 \rangle$  is not in contradiction with the Cosserat-plate model [58] because in the plate case it is automatically zero, since  $\nabla n_0 \equiv 0$ .

<sup>6</sup>In addition, this term is not invariant under reflection across the midsurface, i.e  $\bar{Q}_e = (\bar{Q}_{e,1}, \bar{Q}_{e,2}, \bar{Q}_{e,3}) \mapsto (\bar{Q}_{e,1}, \bar{Q}_{e,2}, -\bar{Q}_{e,3})$  [55].

Thus, our considered form for  $\varrho_m$  and  $\varrho_b$  will be

$$\begin{aligned}\varrho_m^e &= 1 - \frac{\lambda}{\lambda + 2\mu} [\langle \overline{Q}_{e,s}^T (\nabla m | 0) [\nabla_x \Theta(0)]^{-1}, \mathbb{1}_3 \rangle - 2], \\ \varrho_b^e &= - \frac{\lambda}{\lambda + 2\mu} \langle \overline{Q}_{e,s}^T (\nabla (\overline{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3) | 0) [\nabla_x \Theta(0)]^{-1}, \mathbb{1}_3 \rangle \\ &\quad + \frac{\lambda}{\lambda + 2\mu} \langle \overline{Q}_{e,s}^T (\nabla m | 0) [\nabla_x \Theta(0)]^{-1} (\nabla n_0 | 0) [\nabla_x \Theta(0)]^{-1}, \mathbb{1}_3 \rangle.\end{aligned}\tag{4.26}$$

Remark that the condition (3.35) is not satisfied exactly. However, the formula (4.26)<sub>1</sub> has a clear physical significance: in-plane stretch leads to thickness reduction.

Now, our final aim in the determination of  $\varrho_m^e$  and  $\varrho_b^e$  is to compute them for  $\overline{Q}_{e,s} = \mathbb{1}_3$  and  $m = y_0$ , which means the elastic deformation is absent, i.e. we compute

$$\varrho_m^0 = 1 - \frac{\lambda}{\lambda + 2\mu} [\langle (\nabla y_0 | 0) (\nabla y_0 | n_0)^{-1}, \mathbb{1}_3 \rangle - 2],\tag{4.27}$$

$$\begin{aligned}\varrho_b^0 &= - \frac{\lambda}{\lambda + 2\mu} \langle (\nabla (\nabla_x \Theta(0) \cdot e_3) | 0) [\nabla_x \Theta(0)]^{-1}, \mathbb{1}_3 \rangle + \frac{\lambda}{\lambda + 2\mu} \langle (\nabla y_0 | 0) [\nabla_x \Theta(0)]^{-1} (\nabla n_0 | 0) [\nabla_x \Theta(0)]^{-1}, \mathbb{1}_3 \rangle \\ &= - \frac{\lambda}{\lambda + 2\mu} \langle (\nabla n_0 | 0) [\nabla_x \Theta(0)]^{-1}, \mathbb{1}_3 \rangle + \frac{\lambda}{\lambda + 2\mu} \langle (\nabla y_0 | 0) [\nabla_x \Theta(0)]^{-1} (\nabla n_0 | 0) [\nabla_x \Theta(0)]^{-1}, \mathbb{1}_3 \rangle.\end{aligned}\tag{4.28}$$

The identity  $\text{tr}(\mathbf{A}_{y_0}) = \langle (\nabla y_0 | 0) (\nabla y_0 | n_0)^{-1}, \mathbb{1}_3 \rangle = 2$ , (see Proposition A.3) implies that  $\varrho_m^0 = 1$ . Next, we compute  $\varrho_b^0$ . With the help of the curvature tensors  $\mathbf{A}_{y_0}, \mathbf{B}_{y_0}$  (see Proposition A.3) we have

$$\text{tr}[(\nabla y_0 | 0) (\nabla y_0 | n_0)^{-1} (\nabla n_0 | 0) (\nabla y_0 | n_0)^{-1}] = -2\text{H}.\tag{4.29}$$

Hence, we deduce

$$\varrho_b^0 = \frac{\lambda}{(\lambda + 2\mu)} \text{tr}[\mathbf{B}_{y_0}] - \frac{\lambda}{(\lambda + 2\mu)} \text{tr}[\mathbf{L}_{y_0}] = 2 \frac{\lambda}{(\lambda + 2\mu)} \text{H} - 2 \frac{\lambda}{(\lambda + 2\mu)} \text{H} = 0.\tag{4.30}$$

Thus, the reference values  $\varrho_m^0$  and  $\varrho_b^0$  of the parameters  $\varrho_m^e$  and  $\varrho_b^e$  are given by  $\varrho_m^0 = 1$ ,  $\varrho_b^0 = 0$ , which means that in the absence of elastic deformation the ansatz (4.6)  $\varphi_s^0(x_1, x_2, x_3) = y_0(x_1, x_2) + x_3 n_0(x_1, x_2) = \Theta(x_1, x_2, x_3)$  is exact.

### 4.3 The ansatz for the deformation gradient

Having obtained a suitable form of the relevant coefficients  $\varrho_m^e, \varrho_b^e$ , it is expedient to base the expansion of the three-dimensional elastic Cosserat energy on a further simplified expression, please compare with (4.7), namely

$$\begin{aligned}F_s &= \nabla_x \varphi_s(x_1, x_2, x_3) \\ &= (\nabla m | \varrho_m \overline{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0) \cdot e_3) \\ &\quad + x_3 (\nabla [ \underbrace{\overline{Q}_{e,s}(x_1, x_2)}_{\cong \varrho_m^0=1} \nabla_x \Theta(x_1, x_2, 0) \cdot e_3] | \varrho_b \overline{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0) \cdot e_3) \\ &\quad + \frac{x_3^2}{2} (\nabla [ \underbrace{\overline{Q}_{e,s}(x_1, x_2)}_{\cong \varrho_b^0=0} \nabla_x \Theta(x_1, x_2, 0) \cdot e_3] | 0). \\ &\cong (\nabla m | \varrho_m^e \overline{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0) \cdot e_3) \\ &\quad + x_3 (\nabla [ \overline{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0) \cdot e_3] | \varrho_b^e \overline{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0) \cdot e_3).\end{aligned}\tag{4.31}$$

**Remark 4.2.** (Raison d'être)

- 1) The reduced model should at no place contain space derivatives of the thickness stretch  $\varrho_m^e$ , since in the underlying three-dimensional Cosserat model curvature is only present through the dislocation density tensor  $\alpha_\xi$  (or through the wryness tensor  $\Gamma_\xi$ ) related only to rotations  $\overline{Q}_e$ .

- 2) If we blithely use the fully reconstructed deformation gradient  $F_s$  and integrate analytically through the thickness, we would obtain second order derivatives in the energy (through derivatives on  $\varrho_m^e$  and  $\varrho_b^e$ ) both for the midsurface  $m$  and microrotation  $\bar{Q}_e$ , leading to a coupled fourth order problem, a situation which has to be avoided for simplicity and efficiency in a subsequent numerical implementation, taking also in consideration the second order Cosserat bulk problem.
- 3) Keeping the quadratic ansatz (4.7) but neglecting only the derivatives of  $\varrho_m^e$  and  $\varrho_b^e$ , i.e. basing the integration through the thickness instead on (4.7), the reduced ansatz (4.31) would already lead to a second order equilibrium problem and entitle us to skip the quadratic term altogether, since either  $h^5$ -bending terms appear or  $h^3$ - product of membrane and bending appear, which can be dominated through Youngs-inequality by a sum of  $h^2$ -membrane and  $h^4$ -bending terms, which themselves are subordinate (for small  $h$ ) to the already appearing  $h$ -membrane and  $h^3$ -bending terms.
- 4) The error induced by the modified ansatz (4.31) in the energy density will be of higher order under the assumption of small elastic midsurface strain.
- 5) Finally, it should be observed that by using (4.31) we are consistent with John's general result [44, 45] that the stress distribution through the thickness is approximately linear for a thin shell.

Motivated by the above remarks on the ansatz for the (reconstructed) deformation gradient (4.31), the chain rule leads to the approximation

$$\begin{aligned}
F_{s,\xi} &= \nabla_x \varphi_s(x_1, x_2, x_3) [\nabla_x \Theta(x_1, x_2, x_3)]^{-1} \\
&\cong \tilde{F}_{e,s} := (\nabla m | \varrho_m^e \bar{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0) \cdot e_3 | \nabla_x \Theta(x_1, x_2, x_3)]^{-1} \\
&\quad + x_3 (\nabla [\bar{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0) \cdot e_3] | \varrho_b^e \bar{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0) \cdot e_3 | \nabla_x \Theta(x_1, x_2, x_3)]^{-1}.
\end{aligned} \tag{4.32}$$

Our model will be constructed under the following assumption upon the thickness

$$h |\kappa_1| < \frac{1}{2}, \quad h |\kappa_2| < \frac{1}{2}, \tag{4.33}$$

where  $\kappa_1, \kappa_2$  are the principal curvatures.

In consequence, using (iii) from Proposition A.2, we find that the (reconstructed) deformation gradient is given by

$$\begin{aligned}
\tilde{F}_{e,s} &= \frac{1}{b(x_3)} [(\nabla m | \bar{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3 | + x_3 (\nabla [\bar{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3] | 0) + (\varrho_m^e - 1 + x_3 \varrho_b^e)(0 | 0 | \bar{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3)] \\
&\quad \times \left[ \mathbb{1}_3 + x_3 (L_{y_0}^b - 2H \mathbb{1}_3) + x_3^2 K \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] [\nabla_x \Theta(0)]^{-1},
\end{aligned} \tag{4.34}$$

where we have set  $b(x_3) := 1 - 2Hx_3 + Kx_3^2$ . Next, we want to express the tensors

$$\tilde{\mathcal{E}}_s := \bar{U}_{e,s} - \mathbb{1}_3 = \bar{Q}_{e,s}^T \tilde{F}_{e,s} - \mathbb{1}_3, \quad \Gamma_s := (\text{axl}(\bar{Q}_{e,s}^T \partial_{x_1} \bar{Q}_e) | \text{axl}(\bar{Q}_{e,s}^T \partial_{x_2} \bar{Q}_e) | 0) [\nabla_x \Theta(x_3)]^{-1} \tag{4.35}$$

with the help of the usual strain measures in the nonlinear 6-parameter shell theory [36], see Section 6. Therefore, we introduce the following tensor fields on the surface  $\omega_\xi$  [51, 20, 36, 7, 8]

$$\begin{aligned}
\mathcal{E}_{m,s} &:= \bar{Q}_{e,s}^T (\nabla m | \bar{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3 | \nabla_x \Theta(0))^{-1} - \mathbb{1}_3 && \text{(the elastic shell strain tensor),} \\
\mathcal{K}_{e,s} &:= (\text{axl}(\bar{Q}_{e,s}^T \partial_{x_1} \bar{Q}_e) | \text{axl}(\bar{Q}_{e,s}^T \partial_{x_2} \bar{Q}_e) | 0) [\nabla_x \Theta(0)]^{-1} && \text{(elastic shell bending-curvature tensor).}
\end{aligned} \tag{4.36}$$

**Lemma 4.3.** *The following identities are satisfied*

$$\begin{aligned}
i) \quad \mathcal{E}_{m,s} &= (\bar{Q}_{e,s}^T \nabla m - \nabla y_0 | 0) [\nabla_x \Theta(0)]^{-1} = Q_0 (\bar{R}_s^T \nabla m - Q_0^T \nabla y_0 | 0) [\nabla_x \Theta(0)]^{-1}; && ii) \quad \mathcal{E}_{m,s} A_{y_0} = \mathcal{E}_{m,s}; \\
iii) \quad \mathcal{K}_{e,s} A_{y_0} &= \mathcal{K}_{e,s}; && iv) \quad \bar{Q}_{e,s}^T (\nabla [\bar{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3] | 0) [\nabla_x \Theta(0)]^{-1} = C_{y_0} \mathcal{K}_{e,s} - B_{y_0}.
\end{aligned}$$

*Proof.* Starting with i), we observe

$$\mathcal{E}_{m,s} = (\overline{Q}_{e,s}^T \nabla m | n_0) [\nabla_x \Theta(0)]^{-1} - \mathbf{1}_3 = (\overline{Q}_{e,s}^T \nabla m | n_0) [\nabla_x \Theta(0)]^{-1} - (\nabla y_0 | n_0) [\nabla_x \Theta(0)]^{-1}. \quad (4.37)$$

Hence, we obtain ii) and iii) with Proposition A.3 (v).

The last item follows from the same procedure we used to establish (3.22). We have

$$\begin{aligned} \mathcal{K}_{e,s} &= Q_0 \left[ \left( \text{axl}(\overline{R}_s^T \partial_{x_1} \overline{R}_s) | \text{axl}(\overline{R}_s^T \partial_{x_2} \overline{R}_s) | 0 \right) \right. \\ &\quad \left. - \left( \text{axl}(Q_0^T(0) \partial_{x_1} Q_0(0)) | \text{axl}(Q_0^T(0) \partial_{x_2} Q_0(0)) | 0 \right) \right] [\nabla_x \Theta(0)]^{-1}. \end{aligned} \quad (4.38)$$

We compute

$$\begin{aligned} \overline{R}_s^T \partial_{x_\alpha} \overline{R}_s &= (d_1 | d_2 | d_3)^T (\partial_{x_\alpha} d_1 | \partial_{x_\alpha} d_2 | \partial_{x_\alpha} d_3) = \begin{pmatrix} 0 & \langle d_1, \partial_{x_\alpha} d_2 \rangle & \langle d_1, \partial_{x_\alpha} d_3 \rangle \\ \langle d_2, \partial_{x_\alpha} d_1 \rangle & 0 & \langle d_2, \partial_{x_\alpha} d_3 \rangle \\ \langle d_3, \partial_{x_\alpha} d_1 \rangle & \langle d_3, \partial_{x_\alpha} d_2 \rangle & 0 \end{pmatrix}, \\ \text{axl}(\overline{R}_s^T \partial_{x_\alpha} \overline{R}_s) &= \left( -\langle d_2, \partial_{x_\alpha} d_3 \rangle | \langle d_1, \partial_{x_\alpha} d_3 \rangle | -\langle d_1, \partial_{x_\alpha} d_2 \rangle \right)^T, \quad \alpha = 1, 2. \end{aligned} \quad (4.39)$$

Thus, we deduce

$$\begin{aligned} C_{y_0} Q_0(0) &\left( \text{axl}(\overline{R}_s^T (\partial_{x_1} \overline{R}_s) | \text{axl}(\overline{R}_s^T \partial_{x_2} \overline{R}_s) | 0) [\nabla_x \Theta(0)]^{-1} \right. \\ &= Q_0(0) \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\langle d_2, \partial_{x_1} d_3 \rangle & -\langle d_2, \partial_{x_2} d_3 \rangle & 0 \\ \langle d_1, \partial_{x_1} d_3 \rangle & \langle d_1, \partial_{x_2} d_3 \rangle & 0 \\ -\langle d_1, \partial_{x_1} d_2 \rangle & -\langle d_1, \partial_{x_2} d_2 \rangle & 0 \end{pmatrix} [\nabla_x \Theta(0)]^{-1} \\ &= Q_0(0) \overline{R}_s^T (\partial_{x_1} d_3 | \partial_{x_2} d_3 | 0) [\nabla_x \Theta(0)]^{-1} = \overline{Q}_{e,s}^T (\nabla [\overline{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3] | 0) [\nabla_x \Theta(0)]^{-1}. \end{aligned} \quad (4.40)$$

Using Proposition A.3 and (4.38) we conclude

$$C_{y_0} \mathcal{K}_{e,s} = \overline{Q}_{e,s}^T (\nabla [\overline{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3] | 0) [\nabla_x \Theta(0)]^{-1} + B_{y_0} = [\overline{Q}_{e,s}^T (\nabla [\overline{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3] | 0) - \nabla_x \Theta(0) L_{y_0}^b] [\nabla_x \Theta(0)]^{-1}. \quad \blacksquare$$

Accordingly, for the strain tensor corresponding to  $\tilde{\mathcal{E}}_s$  and using Lemma 4.3 we find the following expression of the tensor  $\tilde{\mathcal{E}}_s$  defined by (4.35)

$$\begin{aligned} \tilde{\mathcal{E}}_s &:= \frac{1}{b(x_3)} \left\{ \mathcal{E}_{m,s} + (\varrho_m^e - 1) \overline{Q}_{e,s}^T (0 | 0 | \overline{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3) [\nabla_x \Theta(0)]^{-1} \right. \\ &\quad + x_3 \left[ 2H \mathbf{1}_3 + \overline{Q}_{e,s}^T (\nabla m | \overline{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3) (L_{y_0}^b - 2H \mathbf{1}_3) [\nabla_x \Theta(0)]^{-1} \right. \\ &\quad + C_{y_0} \mathcal{K}_{e,s} - B_{y_0} + (\varrho_m^e - 1) \overline{Q}_{e,s}^T (0 | 0 | \overline{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3) (L_{y_0}^b - 2H \mathbf{1}_3) [\nabla_x \Theta(0)]^{-1} \\ &\quad \left. \left. + \varrho_b^e \overline{Q}_{e,s}^T (0 | 0 | \overline{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3) [\nabla_x \Theta(0)]^{-1} \right] \right. \\ &\quad + x_3^2 \left[ \overline{Q}_{e,s}^T (\nabla m | \overline{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3) K \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} [\nabla_x \Theta(0)]^{-1} \right. \\ &\quad - K \mathbf{1}_3 + \overline{Q}_{e,s}^T (\nabla [\overline{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3] | 0) (L_{y_0}^b - 2H \mathbf{1}_3) [\nabla_x \Theta(0)]^{-1} \\ &\quad + (\varrho_m^e - 1) \overline{Q}_{e,s}^T (0 | 0 | \overline{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3) K \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} [\nabla_x \Theta(0)]^{-1} \\ &\quad \left. \left. + \varrho_b^e \overline{Q}_{e,s}^T (0 | 0 | \overline{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3) (L_{y_0}^b - 2H \mathbf{1}_3) [\nabla_x \Theta(0)]^{-1} \right] \right. \\ &\quad + x_3^3 \left[ \overline{Q}_{e,s}^T (\nabla [\overline{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3] | 0) K \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} [\nabla_x \Theta(0)]^{-1} \right. \\ &\quad \left. \left. + \varrho_b^e \overline{Q}_{e,s}^T (0 | 0 | \overline{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3) K \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} [\nabla_x \Theta(0)]^{-1} \right] \right\}. \end{aligned} \quad (4.41)$$

From Proposition A.3 we have

$$\overline{Q}_{e,s}^T (\nabla m | \overline{Q}_{e,s} \nabla_x \Theta(0).e_3) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} [\nabla_x \Theta(0)]^{-1} = (0|0| \nabla_x \Theta(0).e_3) [\nabla_x \Theta(0)]^{-1} = \mathbb{1}_3 - \mathbf{A}_{y_0}, \quad (4.42)$$

which, together with Lemma 4.3 and  $\mathbf{B}_{y_0} = [\nabla_x \Theta(0)] \mathbf{L}_{y_0}^b [\nabla_x \Theta(0)]^{-1}$  leads to

$$\begin{aligned} \tilde{\mathcal{E}}_s &= \frac{1}{b(x_3)} \left\{ \mathcal{E}_{m,s} + (\varrho_m^e - 1)(0|0| \nabla_x \Theta(0).e_3) [\nabla_x \Theta(0)]^{-1} \right. \\ &\quad + x_3 \left[ 2\mathbf{H} \mathbb{1}_3 + (\mathcal{E}_{m,s} + \mathbb{1}_3) (\mathbf{B}_{y_0} - 2\mathbf{H} \mathbb{1}_3) \right. \\ &\quad \quad \left. + \mathbf{C}_{y_0} \mathcal{K}_{e,s} - \mathbf{B}_{y_0} + [2\mathbf{H} (1 - \varrho_m^e) + \varrho_b^e] (0|0| \nabla_x \Theta(0).e_3) [\nabla_x \Theta(0)]^{-1} \right] \\ &\quad + x_3^2 \left[ \mathbf{K} \mathbb{1}_3 - \mathbf{K} \mathbf{A}_{y_0} - \mathbf{K} \mathbb{1}_3 + [\mathbf{C}_{y_0} \mathcal{K}_{e,s} - \mathbf{B}_{y_0}] (\mathbf{B}_{y_0} - 2\mathbf{H} \mathbb{1}_3) \right. \\ &\quad \quad \left. + (\varrho_m^e - 1)(0|0| \nabla_x \Theta(0).e_3) \mathbf{K} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} [\nabla_x \Theta(0)]^{-1} \right. \\ &\quad \quad \left. + \varrho_b^e (0|0| \nabla_x \Theta(0).e_3) (\mathbf{L}_{y_0}^b - 2\mathbf{H} \mathbb{1}_3) [\nabla_x \Theta(0)]^{-1} \right] \\ &\quad \left. + x_3^3 \left[ 0_3 + \varrho_b^e (0|0| \nabla_x \Theta(0).e_3) \mathbf{K} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} [\nabla_x \Theta(0)]^{-1} \right] \right\}. \end{aligned} \quad (4.43)$$

Further, using the Cayley-Hamilton type equation and item i) from Proposition A.3, item ii) of Lemma 4.3 and  $\mathcal{E}_{m,s} (0|0| \overline{Q}_{e,s} \nabla_x \Theta(0).e_3) = 0_3$  we deduce

$$\begin{aligned} \tilde{\mathcal{E}}_s &= \frac{1}{b(x_3)} \left\{ \mathcal{E}_{m,s} + (\varrho_m^e - 1)(0|0| \nabla_x \Theta(0).e_3) [\nabla_x \Theta(0)]^{-1} \right. \\ &\quad + x_3 \left[ \mathcal{E}_{m,s} (\mathbf{B}_{y_0} - 2\mathbf{H} \mathbf{A}_{y_0}) + \mathbf{C}_{y_0} \mathcal{K}_{e,s} + \mathbf{A}_1 (0|0| \nabla_x \Theta(0).e_3) [\nabla_x \Theta(0)]^{-1} \right] \\ &\quad + x_3^2 \left[ \mathbf{C}_{y_0} \mathcal{K}_{e,s} (\mathbf{B}_{y_0} - 2\mathbf{H} \mathbf{A}_{y_0}) + \mathbf{A}_2 (0|0| \nabla_x \Theta(0).e_3) [\nabla_x \Theta(0)]^{-1} \right] \\ &\quad \left. + x_3^3 \mathbf{K} \varrho_b^e (0|0| \nabla_x \Theta(0).e_3) [\nabla_x \Theta(0)]^{-1} \right\}, \end{aligned} \quad (4.44)$$

where

$$\mathbf{A}_1 := 2\mathbf{H} (1 - \varrho_m^e) + \varrho_b^e, \quad \mathbf{A}_2 := \mathbf{K} (\varrho_m^e - 1) - 2\mathbf{H} \varrho_b^e. \quad (4.45)$$

Using (4.26), we are able to express  $\varrho_m^e$  and  $\varrho_b^e$  also in terms of  $\mathcal{E}_{m,s}, \mathbf{B}_{y_0}, \mathbf{A}_{y_0}, \mathbf{C}_{y_0}$  and  $\mathcal{K}_{e,s}$

$$\begin{aligned} \varrho_m^e &= 1 - \frac{\lambda}{\lambda + 2\mu} [\langle \mathcal{E}_{m,s} + \mathbb{1}_3 - (0|0| \nabla_x \Theta(0).e_3) [\nabla_x \Theta(0)]^{-1}, \mathbb{1}_3 \rangle - 2] = 1 - \frac{\lambda}{\lambda + 2\mu} \text{tr}(\mathcal{E}_{m,s}), \\ \varrho_b^e &= -\frac{\lambda}{\lambda + 2\mu} \langle \mathbf{C}_{y_0} \mathcal{K}_{e,s} - \mathbf{B}_{y_0}, \mathbb{1}_3 \rangle - \frac{\lambda}{\lambda + 2\mu} \langle (\mathcal{E}_{m,s} + \mathbb{1}_3 - (0|0| \nabla_x \Theta(0).e_3) [\nabla_x \Theta(0)]^{-1}) \mathbf{B}_{y_0}, \mathbb{1}_3 \rangle \\ &= -\frac{\lambda}{\lambda + 2\mu} \langle \mathbf{C}_{y_0} \mathcal{K}_{e,s} + \mathcal{E}_{m,s} \mathbf{B}_{y_0}, \mathbb{1}_3 \rangle + \frac{\lambda}{\lambda + 2\mu} \langle (0|0| \nabla_x \Theta(0).e_3) [\nabla_x \Theta(0)]^{-1} \mathbf{B}_{y_0}, \mathbb{1}_3 \rangle \\ &= -\frac{\lambda}{\lambda + 2\mu} \text{tr}[\mathbf{C}_{y_0} \mathcal{K}_{e,s} + \mathcal{E}_{m,s} \mathbf{B}_{y_0}]. \end{aligned} \quad (4.46)$$

Therefore, in view of (4.46) and item ii) of Lemma 4.3, the coefficients  $\mathbf{A}_1$  and  $\mathbf{A}_2$  are given by

$$\begin{aligned} \mathbf{A}_1 &= -\frac{\lambda}{\lambda + 2\mu} \text{tr} \left( \mathcal{E}_{m,s} (\mathbf{B}_{y_0} - 2\mathbf{H} \mathbf{A}_{y_0}) + \mathbf{C}_{y_0} \mathcal{K}_{e,s} \right), \\ \mathbf{A}_2 &= \frac{\lambda}{\lambda + 2\mu} \left[ 2\mathbf{H} \text{tr} (\mathcal{E}_{m,s} \mathbf{B}_{y_0} + \mathbf{C}_{y_0} \mathcal{K}_{e,s}) - \mathbf{K} \text{tr} \mathcal{E}_{m,s} \right]. \end{aligned} \quad (4.47)$$

We also note that

$$(0|0|\nabla_x\Theta(0).e_3)[\nabla_x\Theta(0)]^{-1} = (0|0|n_0)(0|0|n_0)^T = n_0 \otimes n_0. \quad (4.48)$$

In conclusion, the tensor  $\tilde{\mathcal{E}}_s$  defined by (4.35) is completely expressed in terms of  $\mathcal{E}_{m,s}, B_{y_0}, A_{y_0}, C_{y_0}, \mathcal{K}_{e,s}$  and  $n_0 \otimes n_0$ . Similarly, we write  $\Gamma_s$  defined by (4.35) in terms of  $B_{y_0}$  and  $\mathcal{K}_{e,s}$

$$\begin{aligned} \Gamma_s &= \frac{1}{b(x_3)} (\text{axl}(\overline{Q}_{e,s}^T \partial_{x_1} \overline{Q}_e) | \text{axl}(\overline{Q}_{e,s}^T \partial_{x_2} \overline{Q}_{e,s}) | 0) \left[ \mathbb{1}_3 + x_3(L_{y_0}^b - 2H\mathbb{1}_3) + x_3^2 K \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] [\nabla_x\Theta(0)]^{-1} \\ &= \frac{1}{b(x_3)} \mathcal{K}_{e,s} [\nabla_x\Theta(0)] \left[ \mathbb{1}_3 + x_3(L_{y_0}^b - 2H\mathbb{1}_3) \right] [\nabla_x\Theta(0)]^{-1} \\ &\quad + x_3^2 K \frac{1}{b(x_3)} (\text{axl}(\overline{Q}_{e,s}^T \partial_{x_1} \overline{Q}_{e,s}) | \text{axl}(\overline{Q}_{e,s}^T \partial_{x_2} \overline{Q}_{e,s}) | 0) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} [\nabla_x\Theta(0)]^{-1} \\ &= \frac{1}{b(x_3)} \left[ \mathcal{K}_{e,s} + x_3 (\mathcal{K}_{e,s} B_{y_0} - 2H\mathcal{K}_{e,s}) \right]. \end{aligned} \quad (4.49)$$

#### 4.4 Dimensionally reduced energy: analytical integration through the thickness

In what follows, we find the expression of the strain energy density  $W = W_{\text{mp}}(\tilde{\mathcal{E}}_s) + W_{\text{curv}}(\Gamma_s)$  and integrate it over the thickness. To this aim, we introduce the following bilinear forms

$$\begin{aligned} \mathcal{W}_{\text{mp}}(S, T) &= \mu \langle \text{sym } S, \text{sym } T \rangle + \mu_c \langle \text{skew } S, \text{skew } T \rangle + \frac{\lambda}{2} \text{tr}(S) \text{tr}(T) \\ &= \mu \langle \text{dev sym } S, \text{dev sym } T \rangle + \mu_c \langle \text{skew } S, \text{skew } T \rangle + \frac{\kappa}{2} \text{tr}(S) \text{tr}(T), \end{aligned} \quad (4.50)$$

$$\mathcal{W}_{\text{curv}}(S, T) = \mu L_c^2 \left( b_1 \langle \text{dev sym } S, \text{dev sym } T \rangle + b_2 \langle \text{skew } S, \text{skew } T \rangle + 4b_3 \text{tr}(S) \text{tr}(T) \right)$$

for any  $S, T \in \mathbb{R}^{3 \times 3}$ . We remark the identities  $W_{\text{mp}}(S) = \mathcal{W}_{\text{mp}}(S, S)$ ,  $W_{\text{curv}}(S) = \mathcal{W}_{\text{curv}}(S, S)$ . Thus, using (4.44), Lemma 4.3 and the notations (4.47), we obtain

$$\begin{aligned} W_{\text{mp}}(\tilde{\mathcal{E}}_s) &= \frac{1}{b^2(x_3)} W_{\text{mp}} \left( [\mathcal{E}_{m,s} + (\varrho_m^e - 1)n_0 \otimes n_0] + x_3 [(\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) - 2H\mathcal{E}_{m,s} + A_1 n_0 \otimes n_0] \right. \\ &\quad \left. + x_3^2 [C_{y_0} \mathcal{K}_{e,s} B_{y_0} - 2HC_{y_0} \mathcal{K}_{e,s} + A_2 n_0 \otimes n_0] + x_3^3 K \varrho_b^e n_0 \otimes n_0 \right). \end{aligned} \quad (4.51)$$

In order to perform the analytical integration over the thickness, we write  $W_{\text{mp}}(\tilde{\mathcal{E}}_s)$  as a polynomial in  $x_3$  with the coefficients  $C_k$ , i.e.

$$W_{\text{mp}}(\tilde{\mathcal{E}}_s) = \frac{1}{b^2(x_3)} \left( \sum_{k=0}^6 C_k(x_1, x_2) x_3^k \right), \quad (4.52)$$

where

$$\begin{aligned} C_0(x_1, x_2) &= W_{\text{mp}} \left( \mathcal{E}_{m,s} + (\varrho_m^e - 1)n_0 \otimes n_0 \right), \\ C_1(x_1, x_2) &= 2W_{\text{mp}} \left( \mathcal{E}_{m,s} + (\varrho_m^e - 1)n_0 \otimes n_0, (\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) - 2H\mathcal{E}_{m,s} + A_1 n_0 \otimes n_0 \right), \\ C_2(x_1, x_2) &= W_{\text{mp}} \left( (\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) - 2H\mathcal{E}_{m,s} + A_1 n_0 \otimes n_0 \right) \\ &\quad + 2W_{\text{mp}} \left( \mathcal{E}_{m,s} + (\varrho_m^e - 1)n_0 \otimes n_0, C_{y_0} \mathcal{K}_{e,s} B_{y_0} - 2HC_{y_0} \mathcal{K}_{e,s} + A_2 n_0 \otimes n_0 \right), \\ C_3(x_1, x_2) &= 2W_{\text{mp}} \left( \mathcal{E}_{m,s} + (\varrho_m^e - 1)n_0 \otimes n_0, K \varrho_b^e n_0 \otimes n_0 \right) \\ &\quad + 2W_{\text{mp}} \left( (\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) - 2H\mathcal{E}_{m,s} + A_1 n_0 \otimes n_0, C_{y_0} \mathcal{K}_{e,s} B_{y_0} - 2HC_{y_0} \mathcal{K}_{e,s} + A_2 n_0 \otimes n_0 \right), \\ C_4(x_1, x_2) &= W_{\text{mp}} \left( C_{y_0} \mathcal{K}_{e,s} B_{y_0} - 2HC_{y_0} \mathcal{K}_{e,s} + A_2 n_0 \otimes n_0 \right) \\ &\quad + 2W_{\text{mp}} \left( (\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) - 2H\mathcal{E}_{m,s} + A_1 n_0 \otimes n_0, K \varrho_b^e n_0 \otimes n_0 \right), \end{aligned} \quad (4.53)$$

$$\begin{aligned}
C_5(x_1, x_2) &= 2 \mathcal{W}_{\text{mp}} \left( C_{y_0} \mathcal{K}_{e,s} B_{y_0} - 2 \text{H} C_{y_0} \mathcal{K}_{e,s} + A_2 n_0 \otimes n_0, \text{K} \varrho_b^e n_0 \otimes n_0 \right), \\
C_6(x_1, x_2) &= \mathcal{W}_{\text{mp}} \left( \text{K} \varrho_b^e n_0 \otimes n_0 \right).
\end{aligned}$$

Making use of the expansion (since  $x_3 \in (-\frac{h}{2}, \frac{h}{2})$  and  $h$  is small)

$$\begin{aligned}
\frac{1}{b(x_3)} &= \frac{1}{1 - 2 \text{H} x_3 + \text{K} x_3^2} \\
&= 1 + 2 \text{H} x_3 + (4 \text{H}^2 - \text{K}) x_3^2 + (8 \text{H}^3 - 4 \text{H} \text{K}) x_3^3 + (\text{K}^2 - 12 \text{H}^2 \text{K} + 16 \text{H}^4) x_3^4 + O(x_3^5), \quad (4.54)
\end{aligned}$$

and of the relations (3.23) and Proposition A.2 i), the integration can be pursued as follows

$$\begin{aligned}
\int_{\Omega_h} \mathcal{W}_{\text{mp}}(\tilde{\mathcal{E}}_s) \det[\nabla_x \Theta(x)] dV &= \int_{\Omega_h} \left( \sum_{k=0}^6 C_k(x_1, x_2) x_3^k \right) \left[ 1 + 2 \text{H} x_3 + (4 \text{H}^2 - \text{K}) x_3^2 + (8 \text{H}^3 - 4 \text{H} \text{K}) x_3^3 \right. \\
&\quad \left. + (\text{K}^2 - 12 \text{H}^2 \text{K} + 16 \text{H}^4) x_3^4 + O(x_3^5) \right] \det(\nabla y_0 | n_0) dV \\
&= \int_{\omega} \left\{ h C_0 + \frac{h^3}{12} \left[ (4 \text{H}^2 - \text{K}) C_0 + 2 \text{H} C_1 + C_2 \right] \right. \\
&\quad \left. + \frac{h^5}{80} \left[ (\text{K}^2 - 12 \text{H}^2 \text{K} + 16 \text{H}^4) C_0 + (8 \text{H}^3 - 4 \text{H} \text{K}) C_1 \right. \right. \quad (4.55) \\
&\quad \left. \left. + (4 \text{H}^2 - \text{K}) C_2 + 2 \text{H} C_3 + C_4 \right] \right\} \det(\nabla y_0 | n_0) da + O(h^7),
\end{aligned}$$

where  $da = dx_1 dx_2$ .

In view of (4.55), we need to find appropriate expressions for the coefficients  $C_0, C_1, C_2, C_3, C_4$  defined by (4.53). In this line, we designate by  $\mathcal{W}_{\text{shell}}(S, T)$  the bilinear form

$$\begin{aligned}
\mathcal{W}_{\text{shell}}(S, T) &= \mu \langle \text{sym } S, \text{sym } T \rangle + \mu_c \langle \text{skew } S, \text{skew } T \rangle + \frac{\lambda \mu}{\lambda + 2\mu} \text{tr}(S) \text{tr}(T) \\
&= \mu \langle \text{dev sym } S, \text{dev sym } T \rangle + \mu_c \langle \text{skew } S, \text{skew } T \rangle + \frac{2\mu(2\lambda + \mu)}{3(\lambda + 2\mu)} \text{tr}(S) \text{tr}(T), \quad (4.56)
\end{aligned}$$

$$\mathcal{W}_{\text{shell}}(S) = \mathcal{W}_{\text{shell}}(S, S)$$

and we observe that

$$\mathcal{W}_{\text{shell}}(S, T) + \frac{\lambda^2}{2(\lambda + 2\mu)} \text{tr}(S) \text{tr}(T) = \mathcal{W}_{\text{mp}}(S, T), \quad (4.57)$$

since  $\frac{\kappa}{2} - \frac{\lambda^2}{2(\lambda + 2\mu)} = \frac{2\mu(2\lambda + \mu)}{3(\lambda + 2\mu)}$ . Using the notations (4.50), (4.56), we obtain

**Lemma 4.4.** *The following identities*

$$\begin{aligned}
\mathcal{W}_{\text{mp}}(S + \alpha n_0 \otimes n_0, T + \beta n_0 \otimes n_0) &= \mathcal{W}_{\text{mp}}(S, T) + \frac{\lambda}{2} (\alpha \text{tr}(T) + \beta \text{tr}(S)) + \frac{\lambda + 2\mu}{2} \alpha \beta, \\
\mathcal{W}_{\text{mp}}\left(S - \frac{\lambda}{\lambda + 2\mu} (\text{tr}(S)) n_0 \otimes n_0, T + \beta n_0 \otimes n_0\right) &= \mathcal{W}_{\text{shell}}(S, T), \quad (4.58)
\end{aligned}$$

hold true for all tensors  $S, T \in \mathbb{R}^{3 \times 3}$  of the form  $(*|*|0) \cdot [\nabla_x \Theta(0)]^{-1}$  and all  $\alpha, \beta \in \mathbb{R}$ .

*Proof.* In view of the definition (4.50) we see that

$$\begin{aligned}
\mathcal{W}_{\text{mp}}(S + \alpha n_0 \otimes n_0, T + \beta n_0 \otimes n_0) &= \mu \langle \text{sym } S + \alpha n_0 \otimes n_0, \text{sym } T + \beta n_0 \otimes n_0 \rangle \\
&\quad + \mu_c \langle \text{skew } S, \text{skew } T \rangle + \frac{\lambda}{2} (\text{tr}(S) + \alpha) (\text{tr}(T) + \beta). \quad (4.59)
\end{aligned}$$



According to (4.48), we have  $n_0 \otimes n_0 = (0|0|n_0) \cdot [\nabla_x \Theta(0)]^{-1}$ . Since for  $S = (*|*|0) \cdot [\nabla_x \Theta(0)]^{-1}$  we have

$$\begin{aligned} \langle \text{sym } S, n_0 \otimes n_0 \rangle &= \langle (*|*|0) [\nabla_x \Theta(0)]^{-1}, (0|0|n_0) [\nabla_x \Theta(0)]^{-1} \rangle \\ &= \langle (0|0|n_0)^T (*|*|0), [[\nabla_x \Theta(0)]^T \nabla_x \Theta(0)]^{-1} \rangle = \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & 0 \end{pmatrix}, \widehat{\Gamma}_{y_0}^{-1} \right\rangle = 0, \end{aligned} \quad (4.60)$$

we get

$$\begin{aligned} \mathcal{W}_{\text{mp}}(S + \alpha n_0 \otimes n_0, T + \beta n_0 \otimes n_0) &= \mu \langle \text{sym } S, \text{sym } T \rangle + \mu \alpha \beta \langle n_0 \otimes n_0, n_0 \otimes n_0 \rangle + \mu_c \langle \text{skew } S, \text{skew } T \rangle \\ &\quad + \frac{\lambda}{2} \text{tr}(S) \text{tr}(T) + \frac{\lambda}{2} (\alpha \text{tr}(T) + \beta \text{tr}(S)) + \frac{\lambda}{2} \alpha \beta \\ &= \mathcal{W}_{\text{mp}}(S, T) + \frac{\lambda}{2} (\alpha \text{tr}(T) + \beta \text{tr}(S)) + \frac{\lambda + 2\mu}{2} \alpha \beta, \end{aligned} \quad (4.61)$$

which means that the relation (4.58)<sub>1</sub> holds true, for any  $\alpha, \beta \in \mathbb{R}$ .

If we write (4.58)<sub>1</sub> with  $\alpha = \frac{-\lambda}{\lambda + 2\mu} (\text{tr}(S))$ , then we obtain

$$\begin{aligned} \mathcal{W}_{\text{mp}}\left(S - \frac{\lambda}{\lambda + 2\mu} (\text{tr}(S)) n_0 \otimes n_0, T + \beta n_0 \otimes n_0\right) &= \\ &= \mathcal{W}_{\text{mp}}(S, T) + \frac{\lambda}{2} \left( \frac{-\lambda}{\lambda + 2\mu} \text{tr}(S) \text{tr}(T) + \beta \text{tr}(S) \right) + \frac{\lambda + 2\mu}{2} \cdot \frac{-\lambda}{\lambda + 2\mu} \text{tr}(S) \beta \\ &= \mathcal{W}_{\text{mp}}(S, T) - \frac{\lambda^2}{2(\lambda + 2\mu)} \text{tr}(S) \text{tr}(T) = \mathcal{W}_{\text{shell}}(S, T), \end{aligned}$$

where we have used the formula (4.57). Thus, the relation (4.58)<sub>2</sub> is also proved. ■

By virtue of (4.58)<sub>2</sub>, (4.46)<sub>1</sub>, (4.47), (4.36) and Lemma 4.3 we get

$$\begin{aligned} \mathcal{W}_{\text{mp}}\left(\mathcal{E}_{m,s} + (\varrho_m^e - 1)n_0 \otimes n_0, T + \beta n_0 \otimes n_0\right) &= \mathcal{W}_{\text{shell}}(\mathcal{E}_{m,s}, T), \\ \mathcal{W}_{\text{mp}}\left((\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) - 2\text{H} \mathcal{E}_{m,s} + A_1 n_0 \otimes n_0, T + \beta n_0 \otimes n_0\right) &= \mathcal{W}_{\text{shell}}(\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s} - 2\text{H} \mathcal{E}_{m,s}, T), \\ \mathcal{W}_{\text{mp}}\left(C_{y_0} \mathcal{K}_{e,s} B_{y_0} - 2\text{H} C_{y_0} \mathcal{K}_{e,s} + A_2 n_0 \otimes n_0\right) &= \mathcal{W}_{\text{shell}}(C_{y_0} \mathcal{K}_{e,s} B_{y_0} - 2\text{H} C_{y_0} \mathcal{K}_{e,s}) \\ &\quad + \frac{\lambda^2}{2(\lambda + 2\mu)} \left[ \text{tr}((\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) B_{y_0}) \right]^2, \end{aligned} \quad (4.62)$$

where for the last identity we used (4.58)<sub>1</sub> and the relation

$$\mathcal{E}_{m,s} B_{y_0}^2 = 2\text{H} \mathcal{E}_{m,s} B_{y_0} - \text{K} \mathcal{E}_{m,s} A_{y_0} = 2\text{H} \mathcal{E}_{m,s} B_{y_0} - \text{K} \mathcal{E}_{m,s}.$$

From (4.53) and (4.62) we get

$$\begin{aligned} C_0 &= \mathcal{W}_{\text{shell}}(\mathcal{E}_{m,s}), \\ C_1 &= 2\mathcal{W}_{\text{shell}}(\mathcal{E}_{m,s}, \mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s} - 2\text{H} \mathcal{E}_{m,s}) = -4\text{H} \mathcal{W}_{\text{shell}}(\mathcal{E}_{m,s}) + 2\mathcal{W}_{\text{shell}}(\mathcal{E}_{m,s}, \mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}), \\ C_2 &= \mathcal{W}_{\text{shell}}(\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s} - 2\text{H} \mathcal{E}_{m,s}) + 2\mathcal{W}_{\text{shell}}(\mathcal{E}_{m,s}, C_{y_0} \mathcal{K}_{e,s} B_{y_0} - 2\text{H} C_{y_0} \mathcal{K}_{e,s}), \\ C_3 &= 2\mathcal{W}_{\text{shell}}(\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s} - 2\text{H} \mathcal{E}_{m,s}, C_{y_0} \mathcal{K}_{e,s} B_{y_0} - 2\text{H} C_{y_0} \mathcal{K}_{e,s}), \\ C_4 &= \mathcal{W}_{\text{shell}}(C_{y_0} \mathcal{K}_{e,s} B_{y_0} - 2\text{H} C_{y_0} \mathcal{K}_{e,s}) + \frac{\lambda^2}{2(\lambda + 2\mu)} \left[ \text{tr}((\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) B_{y_0}) \right]^2. \end{aligned} \quad (4.63)$$

With the relations (4.63) we can replace the coefficients  $C_0, C_1, C_2, C_3, C_4$  appearing in (4.55) and we obtain

$$\begin{aligned}
(4H^2 - K)C_0 + 2HC_1 + C_2 &= -K W_{\text{shell}}(\mathcal{E}_{m,s}) + W_{\text{shell}}(\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) \\
&\quad + 2W_{\text{shell}}(\mathcal{E}_{m,s}, C_{y_0} \mathcal{K}_{e,s} B_{y_0} - 2HC_{y_0} \mathcal{K}_{e,s}), \\
(K^2 - 12H^2K + 16H^4)C_0 + (8H^3 - 4HK)C_1 + (4H^2 - K)C_2 + 2HC_3 + C_4 \\
&= -K W_{\text{shell}}(\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) + W_{\text{shell}}((\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) B_{y_0}) \\
&\quad + \frac{\lambda^2}{2(\lambda + 2\mu)} \left[ \text{tr}((\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) B_{y_0}) \right]^2 \\
&= -K W_{\text{shell}}(\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) + W_{\text{mp}}((\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) B_{y_0}).
\end{aligned} \tag{4.64}$$

Inserting (4.64) into (4.55) (and neglecting the terms of order  $O(h^7)$ ) we obtain the following result of the integration

$$\begin{aligned}
\int_{\Omega_h} W_{\text{mp}}(\tilde{\mathcal{E}}_s) \det[\nabla_x \Theta(x)] dV &= \int_{\omega} \left[ \left( h - K \frac{h^3}{12} \right) W_{\text{shell}}(\mathcal{E}_{m,s}) + \left( \frac{h^3}{12} - K \frac{h^5}{80} \right) W_{\text{shell}}(\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) \right. \\
&\quad + \frac{h^3}{12} 2W_{\text{shell}}(\mathcal{E}_{m,s}, C_{y_0} \mathcal{K}_{e,s} B_{y_0} - 2HC_{y_0} \mathcal{K}_{e,s}) \\
&\quad \left. + \frac{h^5}{80} W_{\text{mp}}((\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) B_{y_0}) \right] \det(\nabla y_0 | n_0) da.
\end{aligned} \tag{4.65}$$

Remark that using Lemma 4.3 and Proposition A.3 we deduce

$$\begin{aligned}
C_{y_0} \mathcal{K}_{e,s} B_{y_0} - 2HC_{y_0} \mathcal{K}_{e,s} &= C_{y_0} \mathcal{K}_{e,s} (B_{y_0} - 2HA_{y_0}) \\
&= (\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) (B_{y_0} - 2HA_{y_0}) - \mathcal{E}_{m,s} B_{y_0} (B_{y_0} - 2HA_{y_0}) \\
&= (\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) (B_{y_0} - 2HA_{y_0}) + K \mathcal{E}_{m,s}.
\end{aligned} \tag{4.66}$$

Therefore, using again Lemma 4.3 and Proposition A.3, the energy density is rewritten in the following form

$$\begin{aligned}
&\left( h - K \frac{h^3}{12} \right) W_{\text{shell}}(\mathcal{E}_{m,s}) + \left( \frac{h^3}{12} - K \frac{h^5}{80} \right) W_{\text{shell}}(\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) \\
&\quad + \frac{h^3}{12} 2W_{\text{shell}}(\mathcal{E}_{m,s}, C_{y_0} \mathcal{K}_{e,s} B_{y_0} - 2HC_{y_0} \mathcal{K}_{e,s}) + \frac{h^5}{80} W_{\text{mp}}((\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) B_{y_0}) \\
&= \left( h + K \frac{h^3}{12} \right) W_{\text{shell}}(\mathcal{E}_{m,s}) + \left( \frac{h^3}{12} - K \frac{h^5}{80} \right) W_{\text{shell}}(\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) \\
&\quad + \frac{h^3}{12} 2W_{\text{shell}}(\mathcal{E}_{m,s}, (\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) (B_{y_0} - 2HA_{y_0})) \\
&\quad + \frac{h^5}{80} W_{\text{mp}}((\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) B_{y_0}) \\
&= \left( h + K \frac{h^3}{12} \right) W_{\text{shell}}(\mathcal{E}_{m,s}) + \left( \frac{h^3}{12} - K \frac{h^5}{80} \right) W_{\text{shell}}(\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) \\
&\quad - \frac{h^3}{3} H W_{\text{shell}}(\mathcal{E}_{m,s}, \mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) + \frac{h^3}{6} W_{\text{shell}}(\mathcal{E}_{m,s}, (\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) B_{y_0}) \\
&\quad + \frac{h^5}{80} W_{\text{mp}}((\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) B_{y_0}).
\end{aligned} \tag{4.67}$$

Analogously, using (4.49), we integrate the curvature part of the strain energy density

$$\begin{aligned}
\int_{\Omega_h} W_{\text{curv}}(\Gamma_s) \det[\nabla_x \Theta(x)] dV &= \int_{\Omega_h} W_{\text{curv}}(\mathcal{K}_{e,s} + x_3 (\mathcal{K}_{e,s} B_{y_0} - 2H \mathcal{K}_{e,s})) \frac{\det(\nabla y_0 | n_0)}{b(x_3)} dV \\
&= \int_{\Omega_h} \left( D_0 + D_1 x_3 + D_2 x_3^2 \right) \left[ 1 + 2H x_3 + (4H^2 - K) x_3^2 + (8H^3 - 4HK) x_3^3 \right. \\
&\quad \left. + (K^2 - 12H^2K + 16H^4) x_3^4 + O(x_3^5) \right] \det(\nabla y_0 | n_0) dV \\
&= \int_{\omega} \left\{ h D_0 + \frac{h^3}{12} \left[ (4H^2 - K) D_0 + 2H D_1 + D_2 \right] + \frac{h^5}{80} \left[ (K^2 - 12H^2K + 16H^4) D_0 \right. \right. \\
&\quad \left. \left. + (8H^3 - 4HK) D_1 + (4H^2 - K) D_2 \right] \right\} \det(\nabla y_0 | n_0) da + O(h^7),
\end{aligned} \tag{4.68}$$

where we have denoted by  $D_k$  the coefficients of  $x_3^k$  ( $k = 0, 1, 2$ ) in the expression

$$\begin{aligned} W_{\text{curv}}(\mathcal{K}_{e,s} + x_3(\mathcal{K}_{e,s}\mathbf{B}_{y_0} - 2\mathbf{H}\mathcal{K}_{e,s})) &= D_0(x_1, x_2) + D_1(x_1, x_2)x_3 + D_2(x_1, x_2)x_3^2, \quad \text{with} \quad (4.69) \\ D_0 &= W_{\text{curv}}(\mathcal{K}_{e,s}), \quad D_1 = 2\mathcal{W}_{\text{curv}}(\mathcal{K}_{e,s}, \mathcal{K}_{e,s}\mathbf{B}_{y_0} - 2\mathbf{H}\mathcal{K}_{e,s}), \quad D_2 = W_{\text{curv}}(\mathcal{K}_{e,s}\mathbf{B}_{y_0} - 2\mathbf{H}\mathcal{K}_{e,s}). \end{aligned}$$

We write the coefficients of  $\frac{h^3}{12}$  and  $\frac{h^5}{80}$  in (4.68) with the help of (4.69)<sub>2,3,4</sub> as follows

$$\begin{aligned} (4\mathbf{H}^2 - \mathbf{K})D_0 + 2\mathbf{H}D_1 + D_2 &= -\mathbf{K}W_{\text{curv}}(\mathcal{K}_{e,s}) + W_{\text{curv}}(\mathcal{K}_{e,s}\mathbf{B}_{y_0}), \\ (\mathbf{K}^2 - 12\mathbf{H}^2\mathbf{K} + 16\mathbf{H}^4)D_0 + (8\mathbf{H}^3 - 4\mathbf{H}\mathbf{K})D_1 + (4\mathbf{H}^2 - \mathbf{K})D_2 &= \\ &= \mathbf{K}^2W_{\text{curv}}(\mathcal{K}_{e,s}) - 4\mathbf{H}\mathbf{K}\mathcal{W}_{\text{curv}}(\mathcal{K}_{e,s}, \mathcal{K}_{e,s}\mathbf{B}_{y_0}) + (4\mathbf{H}^2 - \mathbf{K})W_{\text{curv}}(\mathcal{K}_{e,s}\mathbf{B}_{y_0}) \\ &= -\mathbf{K}W_{\text{curv}}(\mathcal{K}_{e,s}\mathbf{B}_{y_0}) + W_{\text{curv}}(\mathcal{K}_{e,s}\mathbf{B}_{y_0}^2). \end{aligned} \quad (4.70)$$

Inserting (4.70) into (4.68) (and again neglecting the terms of order  $O(h^7)$ ) we arrive at the following result of this integration

$$\begin{aligned} \int_{\Omega_h} W_{\text{curv}}(\Gamma_s) \det[\nabla_x \Theta(x)] dV &= \\ &= \int_{\omega} \left[ \left( h - \mathbf{K} \frac{h^3}{12} \right) W_{\text{curv}}(\mathcal{K}_{e,s}) + \left( \frac{h^3}{12} - \mathbf{K} \frac{h^5}{80} \right) W_{\text{curv}}(\mathcal{K}_{e,s}\mathbf{B}_{y_0}) + \frac{h^5}{80} W_{\text{curv}}(\mathcal{K}_{e,s}\mathbf{B}_{y_0}^2) \right] \det(\nabla y_0|n_0) da. \end{aligned} \quad (4.71)$$

In order to write the external loads potential in the shell model, we perform next the integration over the thickness of the relations (3.25). Thus, from (3.26) and (4.6) we find

$$\begin{aligned} \tilde{u}(x_i) &= \varphi(x_i) - \Theta(x_i) = \left( m + x_3 \varrho_m \bar{Q}_{e,s} \cdot n_0 + \frac{x_3^2}{2} \varrho_b \bar{Q}_{e,s} \cdot n_0 \right) - (y_0 + x_3 n_0) \\ &= (m - y_0) + x_3 (\varrho_m \bar{Q}_{e,s} \cdot n_0 - n_0) + \frac{x_3^2}{2} \varrho_b \bar{Q}_{e,s} \cdot n_0. \end{aligned}$$

We insert this into (3.25)<sub>1</sub> and use the approximation  $\varrho_m \cong \varrho_m^0 = 1$ ,  $\varrho_b \cong \varrho_b^0 = 0$  as in (4.26) to obtain the simplified form

$$\int_{\Omega_h} \langle \tilde{f}, \tilde{u} \rangle dV = \int_{\omega} \left( \langle \int_{-h/2}^{h/2} \tilde{f} dx_3, m - y_0 \rangle + \langle \int_{-h/2}^{h/2} x_3 \tilde{f} dx_3, (\bar{Q}_{e,s} - \mathbf{1}_3) \cdot n_0 \rangle \right) da. \quad (4.72)$$

Denoting with  $\tilde{t}^{\pm}(x_1, x_2) := \tilde{t}(x_1, x_2, \pm \frac{h}{2})$  and taking into account that  $\Gamma_t = \left( \omega \times \left\{ \frac{h}{2} \right\} \right) \cup \left( \omega \times \left\{ -\frac{h}{2} \right\} \right) \cup \left( \gamma_t \times \left( -\frac{h}{2}, \frac{h}{2} \right) \right)$ , we obtain similarly

$$\begin{aligned} \int_{\Gamma_t} \langle \tilde{t}, \tilde{u} \rangle dS &= \int_{\omega} \langle \tilde{t}^{\pm}, (m - y_0) \pm \frac{h}{2} (\varrho_m \bar{Q}_{e,s} \cdot n_0 - n_0) + \frac{h^2}{8} \varrho_b \bar{Q}_{e,s} \cdot n_0 \rangle da \\ &\quad + \int_{\gamma_t} \int_{-h/2}^{h/2} \langle \tilde{t}, (m - y_0) + x_3 (\varrho_m \bar{Q}_{e,s} \cdot n_0 - n_0) + \frac{x_3^2}{2} \varrho_b \bar{Q}_{e,s} \cdot n_0 \rangle dx_3 ds. \end{aligned}$$

Using the same approximation as before ( $\varrho_m \cong \varrho_m^0 = 1$ ,  $\varrho_b \cong \varrho_b^0 = 0$ , see (4.26)) we find

$$\begin{aligned} \int_{\Gamma_t} \langle \tilde{t}, \tilde{u} \rangle dS &= \int_{\omega} \langle \tilde{t}^+ + \tilde{t}^-, m - y_0 \rangle da + \int_{\omega} \langle \frac{h}{2} (\tilde{t}^+ - \tilde{t}^-), (\bar{Q}_{e,s} - \mathbf{1}_3) \cdot n_0 \rangle da \\ &\quad + \int_{\gamma_t} \langle \int_{-h/2}^{h/2} \tilde{t} dx_3, m - y_0 \rangle ds + \int_{\gamma_t} \langle \int_{-h/2}^{h/2} x_3 \tilde{t} dx_3, (\bar{Q}_{e,s} - \mathbf{1}_3) \cdot n_0 \rangle ds, \end{aligned} \quad (4.73)$$

where  $ds$  is the arclength element along the curve  $\gamma_t$  and  $da = dx_1 dx_2$ . With (4.72) and (4.73), the potential of external applied loads  $\bar{\Pi}(m, \bar{Q}_{e,s}) = \bar{\Pi}(\varphi, \bar{R})$  in (3.25) can be written in the form

$$\bar{\Pi}(m, \bar{Q}_{e,s}) = \Pi_{\omega}(m, \bar{Q}_{e,s}) + \Pi_{\gamma_t}(m, \bar{Q}_{e,s}), \quad (4.74)$$

with

$$\Pi_\omega(m, \bar{Q}_{e,s}) = \int_\omega \langle \bar{f}, \bar{u} \rangle da + \Lambda_\omega(\bar{Q}_{e,s}), \quad \Pi_{\gamma_t}(m, \bar{Q}_{e,s}) = \int_{\gamma_t} \langle \bar{t}, \bar{u} \rangle ds + \Lambda_{\gamma_t}(\bar{Q}_{e,s}), \quad (4.75)$$

where  $\bar{u}(x_1, x_2) = m(x_1, x_2) - y_0(x_1, x_2)$  is the displacement vector of the midsurface and

$$\begin{aligned} \bar{f} &= \int_{-h/2}^{h/2} \tilde{f} dx_3 + (\tilde{t}^+ + \tilde{t}^-), & \bar{t} &= \int_{-h/2}^{h/2} \tilde{t} dx_3, \\ \Lambda_\omega(\bar{Q}_{e,s}) &= \int_\omega \left\langle \int_{-h/2}^{h/2} x_3 \tilde{f} dx_3 + \frac{h}{2}(\tilde{t}^+ - \tilde{t}^-), (\bar{Q}_{e,s} - \mathbb{1}_3) \cdot n_0 \right\rangle da + \bar{\Pi}_\omega(\bar{Q}_{e,s}), \\ \Lambda_{\gamma_t}(\bar{Q}_{e,s}) &= \int_{\gamma_t} \left\langle \int_{-h/2}^{h/2} x_3 \tilde{t} dx_3, (\bar{Q}_{e,s} - \mathbb{1}_3) \cdot n_0 \right\rangle ds + \bar{\Pi}_{\gamma_t}(\bar{Q}_{e,s}) \end{aligned} \quad (4.76)$$

and  $\bar{\Pi}_\omega(\bar{Q}_{e,s}) = \tilde{\Pi}_{\Omega_h}(\bar{R})$ ,  $\bar{\Pi}_{\gamma_t}(\bar{Q}_{e,s}) = \tilde{\Pi}_{\Gamma_t}(\bar{R})$ , since  $\bar{R}$  is independent of  $x_3$ .

The functions  $\bar{\Pi}_\omega, \bar{\Pi}_{\gamma_t} : L^2(\omega, \text{SO}(3)) \rightarrow \mathbb{R}$  are assumed to be continuous and bounded operators.

## 5 The new geometrically nonlinear Cosserat shell model

### 5.1 Formulation of the minimization problem

Gathering our results, see (3.23), (4.65), (4.67) and (4.71), we have obtained the following two-dimensional minimization problem for the deformation of the midsurface  $m : \omega \rightarrow \mathbb{R}^3$  and the microrotation of the shell  $\bar{Q}_{e,s} : \omega \rightarrow \text{SO}(3)$  solving on  $\omega \subset \mathbb{R}^2$ : minimize with respect to  $(m, \bar{Q}_{e,s})$  the functional

$$I = \int_\omega \left[ W_{\text{memb}}(\mathcal{E}_{m,s}) + W_{\text{memb,bend}}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) + W_{\text{bend,curv}}(\mathcal{K}_{e,s}) \right] \det(\nabla y_0 | n_0) da - \bar{\Pi}(m, \bar{Q}_{e,s}), \quad (5.1)$$

where the membrane part  $W_{\text{memb}}(\mathcal{E}_{m,s})$ , the membrane–bending part  $W_{\text{memb,bend}}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s})$  and the bending–curvature part  $W_{\text{bend,curv}}(\mathcal{K}_{e,s})$  of the shell energy density are given by

$$\begin{aligned} W_{\text{memb}}(\mathcal{E}_{m,s}) &= \left( h + K \frac{h^3}{12} \right) W_{\text{shell}}(\mathcal{E}_{m,s}), \\ W_{\text{memb,bend}}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) &= \left( \frac{h^3}{12} - K \frac{h^5}{80} \right) W_{\text{shell}}(\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) \\ &\quad - \frac{h^3}{3} H W_{\text{shell}}(\mathcal{E}_{m,s}, \mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) + \frac{h^3}{6} W_{\text{shell}}(\mathcal{E}_{m,s}, (\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) B_{y_0}) \\ &\quad + \frac{h^5}{80} W_{\text{mp}}((\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) B_{y_0}), \\ W_{\text{bend,curv}}(\mathcal{K}_{e,s}) &= \left( h - K \frac{h^3}{12} \right) W_{\text{curv}}(\mathcal{K}_{e,s}) + \left( \frac{h^3}{12} - K \frac{h^5}{80} \right) W_{\text{curv}}(\mathcal{K}_{e,s} B_{y_0}) + \frac{h^5}{80} W_{\text{curv}}(\mathcal{K}_{e,s} B_{y_0}^2) \end{aligned} \quad (5.2)$$

and

$$\begin{aligned} \mathcal{E}_{m,s} &= \bar{Q}_{e,s}^T \tilde{F}_m - \mathbb{1}_3, & \tilde{F}_m &= (\nabla m | \bar{Q}_{e,s} \nabla_x \Theta(0) \cdot e_3) [\nabla_x \Theta(0)]^{-1}, \\ \mathcal{K}_{e,s} &= (\text{axl}(\bar{Q}_{e,s}^T \partial_{x_1} \bar{Q}_{e,s}) | \text{axl}(\bar{Q}_{e,s}^T \partial_{x_2} \bar{Q}_{e,s}) | 0) [\nabla_x \Theta(0)]^{-1}, \\ \Theta(x_1, x_2, x_3) &= y_0(x_1, x_2) + x_3 n_0(x_1, x_2), & \nabla_x \Theta(0) &= (\nabla y_0 | n_0), & n_0 &= (\nabla_x \Theta(0)) \cdot e_3 \\ B_{y_0} &= -(\nabla n_0 | 0) [\nabla_x \Theta(0)]^{-1}, & C_{y_0} &= \det(\nabla_x \Theta(0)) [\nabla_x \Theta(0)]^{-T} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} [\nabla_x \Theta(0)]^{-1} \in \mathfrak{so}(3), \\ K &= \det(L_{y_0}), & 2H &= \text{tr}(L_{y_0}), & L_{y_0} &= -([\nabla y_0]^T \nabla y_0)^{-1} ([\nabla n_0]^T \nabla y_0), \\ W_{\text{shell}}(S) &= \mu \|\text{sym } S\|^2 + \mu_c \|\text{skew } S\|^2 + \frac{\lambda \mu}{\lambda + 2\mu} [\text{tr}(S)]^2, \\ W_{\text{shell}}(S, T) &= \mu \langle \text{sym } S, \text{sym } T \rangle + \mu_c \langle \text{skew } S, \text{skew } T \rangle + \frac{\lambda \mu}{\lambda + 2\mu} \text{tr}(S) \text{tr}(T), \end{aligned} \quad (5.3)$$

$$\begin{aligned}
W_{\text{mp}}(S) &= \mu \|\text{sym } S\|^2 + \mu_c \|\text{skew } S\|^2 + \frac{\lambda}{2} [\text{tr}(S)]^2, \\
W_{\text{curv}}(S) &= \mu L_c^2 \left( b_1 \|\text{dev sym } S\|^2 + b_2 \|\text{skew } S\|^2 + 4 b_3 [\text{tr}(S)]^2 \right).
\end{aligned}$$

In this formulation, all the constitutive coefficients are deduced from the three-dimensional formulation, without using any a posteriori fitting of some two-dimensional constitutive coefficients.

The potential of applied external loads  $\bar{\Pi}(m, \bar{Q}_{e,s})$  appearing in (5.1) is expressed by the relations (4.74), (4.76).

We consider the following boundary conditions for the midsurface deformation  $m$  and rotation field  $\bar{R}_s$  on the Dirichlet part of the lateral boundary  $\gamma_0 \subset \partial\omega$ :

$$m|_{\gamma_0} = m_0, \quad \text{simply supported (fixed, welded)} \quad \bar{R}_s|_{\gamma_0} = \hat{R}, \quad \text{(clamped)}.$$

**Remark 5.1.** *Summarizing, the present shell model was derived:*

- under the assumption that  $h|\kappa_1| < \frac{1}{2}$ ,  $h|\kappa_2| < \frac{1}{2}$
- considering an approximation of the elastic rotation  $\bar{Q}_e : \Omega_h \rightarrow \text{SO}(3)$

$$\bar{Q}_e(x_1, x_2, x_3) \cong \bar{Q}_{e,s}(x_1, x_2) = \bar{R}_s(x_1, x_2) Q_0^T(x_1, x_2, 0); \quad (5.4)$$

- choosing an 8-parameter quadratic ansatz in the thickness direction for the reconstructed total deformation  $\varphi_s : \Omega_h \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$  of the shell-like structure

$$\varphi_s(x_1, x_2, x_3) = m(x_1, x_2) + \left( x_3 \varrho_m(x_1, x_2) + \frac{x_3^2}{2} \varrho_b(x_1, x_2) \right) \bar{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0) \cdot e_3; \quad (5.5)$$

- taking the exact form of  $\varrho_m$  and considering a suitable approximation for  $\varrho_b$  (coming from a generalized plane stress condition)

$$\begin{aligned}
\varrho_m &= 1 - \frac{\lambda}{\lambda + 2\mu} [\langle \bar{Q}_{e,s}^T (\nabla m|_0) [\nabla_x \Theta(0)]^{-1}, \mathbf{1}_3 \rangle - 2] =: \varrho_m^e, \\
\varrho_b &= \frac{\lambda}{\lambda + 2\mu} \langle \bar{Q}_{e,s}^T (\nabla m|_0) [\nabla_x \Theta(0)]^{-1} (\nabla n_0|_0) [\nabla_x \Theta(0)]^{-1}, \mathbf{1}_3 \rangle \\
&\quad - \frac{\lambda}{\lambda + 2\mu} \langle \bar{Q}_{e,s}^T (\nabla (\bar{Q}_{e,s} \nabla_x \Theta(0)) \cdot e_3)|_0 [\nabla_x \Theta(0)]^{-1}, \mathbf{1}_3 \rangle \\
&\quad + \frac{\lambda^2}{(\lambda + 2\mu)^2} \left[ \langle \bar{Q}_{e,s}^T (\nabla (\bar{Q}_{e,s} \nabla_x \Theta(0)) \cdot e_3)|_0 [\nabla_x \Theta(0)]^{-1}, \mathbf{1}_3 \rangle \right] \left[ \langle \bar{Q}_{e,s}^T (\nabla m|_0) [\nabla_x \Theta(0)]^{-1}, \mathbf{1}_3 \rangle - 2 \right] \\
&\cong \varrho_b^e := \frac{\lambda}{\lambda + 2\mu} \langle \bar{Q}_{e,s}^T (\nabla m|_0) [\nabla_x \Theta(0)]^{-1} (\nabla n_0|_0) [\nabla_x \Theta(0)]^{-1}, \mathbf{1}_3 \rangle \\
&\quad - \frac{\lambda}{\lambda + 2\mu} \langle \bar{Q}_{e,s}^T (\nabla (\bar{Q}_{e,s} \nabla_x \Theta(0)) \cdot e_3)|_0 [\nabla_x \Theta(0)]^{-1}, \mathbf{1}_3 \rangle;
\end{aligned} \quad (5.6)$$

- choosing a further approximation of the deformation gradient (by neglecting space derivatives of  $\varrho_m^e$  and  $\varrho_b^e$ , respectively)

$$\begin{aligned}
F_s &= \nabla_x \varphi_s(x_1, x_2, x_3) = (\nabla m|_0 \varrho_m \bar{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0) \cdot e_3) \\
&\quad + x_3 (\nabla [\varrho_m \bar{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0) \cdot e_3] | \varrho_b \bar{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0) \cdot e_3) \\
&\quad + \frac{x_3^2}{2} (\nabla [\varrho_b \bar{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0) \cdot e_3] |_0) \\
&\cong \tilde{F}_s := (\nabla m|_0 \varrho_m^e \bar{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0) \cdot e_3) \\
&\quad + x_3 (\nabla [\bar{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0) \cdot e_3] | \varrho_b^e \bar{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0) \cdot e_3),
\end{aligned} \quad (5.7)$$

and therefore, the following approximation of the reconstructed gradient

$$\begin{aligned} F_{s,\xi} &= \nabla_x \varphi_s(x_1, x_2, x_3) [\nabla_x \Theta(x_1, x_2, x_3)]^{-1} \\ &\cong \widetilde{F}_{e,s} := (\nabla m | \varrho_m^e \overline{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0) \cdot e_3) [\nabla_x \Theta(x_1, x_2, x_3)]^{-1} \\ &\quad + x_3 (\nabla [\overline{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0) \cdot e_3] | \varrho_b^e \overline{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0) \cdot e_3) [\nabla_x \Theta(x_1, x_2, x_3)]^{-1}. \end{aligned} \quad (5.8)$$

Moreover, we have used the full expressions of  $[\nabla_x \Theta(x_3)]^{-1}$  and  $\det(\nabla_x \Theta(x_3))$

$$\begin{aligned} [\nabla_x \Theta(x_3)]^{-1} &= \frac{1}{1 - 2Hx_3 + Kx_3^2} \left[ \mathbb{1}_3 + x_3(L_{y_0}^b - 2H\mathbb{1}_3) + x_3^2 K \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] [\nabla_x \Theta(0)]^{-1}, \\ \det(\nabla_x \Theta(x_3)) &= \det(\nabla_x \Theta(0)) \left[ 1 - 2x_3 H + x_3^2 K \right]. \end{aligned} \quad (5.9)$$

- neglecting the terms of order  $O(h^7)$  in the final form of the energy.

After a shell model is proposed, there is a basic requirement: the 2D-shell model must be invariant w.r.t. a reparametrization of the midsurface coordinates. Here, the total elastically stored energy

$$W_{\text{memb}}(\mathcal{E}_{m,s}) + W_{\text{memb,bend}}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) + W_{\text{bend,curv}}(\mathcal{K}_{e,s})$$

depends on the midsurface deformation gradient  $\nabla m$  and microrotations  $\overline{Q}_{e,s}$  together with their space derivatives only through the frame-indifferent measure tensors  $\mathcal{E}_{m,s}, \mathcal{K}_{e,s}, B_{y_0}$  and  $C_{y_0}$ , which are invariant to the reparametrization.

## 5.2 Consistency with the Cosserat plate model

In the case of Cosserat plates we have  $\Theta(x_1, x_2, x_3) = (x_1, x_2, x_3)$  and

$$\begin{aligned} \nabla_x \Theta(x_3) &= \mathbb{1}_3, & y_0(x_1, x_2) &= (x_1, x_2) =: \text{id}(x_1, x_2), & Q_0 &= \mathbb{1}_3, & n_0 &= e_3, & d_i^0 &= e_i, \\ B_{\text{id}} &= 0_3, & C_{\text{id}} &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(3), & L_{\text{id}} &= 0_2, & K &= 0, & H &= 0. \end{aligned} \quad (5.10)$$

Therefore, for the Cosserat plate model the minimization problem reads: find the deformation of the midsurface  $m : \omega \rightarrow \mathbb{R}^3$  and the microrotation of the shell  $\overline{Q}_{e,s} : \omega \rightarrow \text{SO}(3)$  solving on  $\omega \subset \mathbb{R}^2$ :

$$I = \int_{\omega} \left[ W_{\text{memb}}(\mathcal{E}_{m,s}) + W_{\text{memb,bend}}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) + W_{\text{bend,curv}}(\mathcal{K}_{e,s}) \right] da \rightarrow \min. \text{ w.r.t. } (m, \overline{Q}_{e,s}) \quad (5.11)$$

where the membrane part  $W_{\text{memb}}(\mathcal{E}_{m,s})$ , the membrane-bending part  $W_{\text{memb,bend}}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s})$  and the bending-curvature part  $W_{\text{bend,curv}}(\mathcal{K}_{e,s})$  of the shell energy density are given by

$$\begin{aligned} W_{\text{memb}}(\mathcal{E}_{m,s}) &= h W_{\text{shell}}(\mathcal{E}_{m,s}), & W_{\text{memb,bend}}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) &= \frac{h^3}{12} W_{\text{shell}}(C_{\text{id}} \mathcal{K}_{e,s}), \\ W_{\text{bend,curv}}(\mathcal{K}_{e,s}) &= h W_{\text{curv}}(\mathcal{K}_{e,s}), \end{aligned} \quad (5.12)$$

and

$$\begin{aligned} \mathcal{E}_{m,s} &= \overline{Q}_{e,s}^T \widetilde{F}_m - \mathbb{1}_3, & \widetilde{F}_m &= (\nabla m | \overline{Q}_{e,s} \cdot e_3), \\ \mathcal{K}_{e,s} &= (\text{axl}(\overline{Q}_{e,s}^T \partial_{x_1} \overline{Q}_{e,s}) | \text{axl}(\overline{Q}_{e,s}^T \partial_{x_2} \overline{Q}_{e,s}) | 0), & C_{\text{id}} \mathcal{K}_{e,s} &= \overline{Q}_{e,s}^T (\nabla[\overline{Q}_{e,s} \cdot e_3] | 0), \\ W_{\text{shell}}(S) &= \mu \|\text{sym } S\|^2 + \mu_c \|\text{skew } S\|^2 + \frac{\lambda \mu}{\lambda + 2\mu} [\text{tr } S]^2, \\ W_{\text{curv}}(S) &= \mu L_c^2 (b_1 \|\text{dev sym } S\|^2 + b_2 \|\text{skew } S\|^2 + 4b_3 [\text{tr}(S)]^2). \end{aligned} \quad (5.13)$$

In view of Lemma 4.3 the following identity is satisfied

$$\overline{Q}_{e,s}^T (\nabla[\overline{Q}_{e,s} \cdot e_3] | 0) = C_{\text{id}} \mathcal{K}_{e,s}. \quad (5.14)$$

Hence,  $C_{\text{id}} \mathcal{K}_{e,s}$  coincides with the second order *non-symmetric bending tensor*

$$\mathfrak{K}_b := \overline{Q}_{e,s}^T (\nabla[\overline{Q}_{e,s} \cdot e_3] | 0) + B_{y_0} = \overline{Q}_{e,s}^T (\nabla[\overline{Q}_{e,s} \cdot e_3] | 0)$$

considered by Neff in [58]. In consequence, the particular case considered in this subsection is the geometrically nonlinear Cosserat-shell model including size effects introduced in [58] and then used in a numerical approach by Sander et al. [82], but corresponds to another representation of the curvature energy from the three-dimensional formulation.

**Remark 5.2.** *This Cosserat plate model was derived:*

- *considering an approximation of the elastic rotation  $\overline{Q}_e : \Omega_h \rightarrow \text{SO}(3)$*

$$\overline{Q}_e(x_1, x_2, x_3) \cong \overline{Q}_{e,s}(x_1, x_2); \quad (5.15)$$

- *choosing an 8-parameter quadratic ansatz in the thickness direction for the reconstructed total deformation  $\varphi_s : \Omega_h \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$  of the shell-like structure*

$$\varphi_s(x_1, x_2, x_3) = m(x_1, x_2) + \left( x_3 \varrho_m(x_1, x_2) + \frac{x_3^2}{2} \varrho_b(x_1, x_2) \right) \overline{Q}_{e,s}(x_1, x_2) \cdot e_3; \quad (5.16)$$

- *taking the exact form of  $\varrho_m$  and considering a suitable approximation for  $\varrho_b$  (coming from a generalized plane stress condition)*

$$\begin{aligned} \varrho_m &= 1 - \frac{\lambda}{\lambda + 2\mu} [\langle \overline{Q}_{e,s}^T (\nabla m | 0), \mathbb{1}_3 \rangle - 2] =: \varrho_m^e, \\ \varrho_b &= -\frac{\lambda}{\lambda + 2\mu} \langle \overline{Q}_{e,s}^T (\nabla(\overline{Q}_{e,s} \cdot e_3) | 0), \mathbb{1}_3 \rangle \\ &\quad + \frac{\lambda^2}{(\lambda + 2\mu)^2} \left[ \langle \overline{Q}_{e,s}^T (\nabla(\overline{Q}_{e,s} \cdot e_3) | 0), \mathbb{1}_3 \rangle \right] \left[ \langle \overline{Q}_{e,s}^T (\nabla m | 0), \mathbb{1}_3 \rangle - 2 \right] \\ &\cong \varrho_b^e := -\frac{\lambda}{\lambda + 2\mu} \langle \overline{Q}_{e,s}^T (\nabla(\overline{Q}_{e,s} \cdot e_3) | 0), \mathbb{1}_3 \rangle; \end{aligned}$$

- *choosing a further approximation of the deformation gradient (by neglecting space derivatives of  $\varrho_m^e$  and  $\varrho_b^e$ , respectively)*

$$\begin{aligned} F_s &= (\nabla m | \varrho_m \overline{Q}_{e,s}(x_1, x_2) \cdot e_3) + x_3 (\nabla [\varrho_m \overline{Q}_{e,s}(x_1, x_2) \cdot e_3] | \varrho_b \overline{Q}_{e,s}(x_1, x_2) \cdot e_3) \\ &\quad + \frac{x_3^2}{2} (\nabla [\varrho_b \overline{Q}_{e,s}(x_1, x_2) \cdot e_3] | 0) \\ &\cong \tilde{F}_s := (\nabla m | \varrho_m^e \overline{Q}_{e,s}(x_1, x_2) \cdot e_3) + x_3 (\nabla [\overline{Q}_{e,s}(x_1, x_2) \cdot e_3] | \varrho_b^e \overline{Q}_{e,s}(x_1, x_2) \cdot e_3); \end{aligned} \quad (5.17)$$

- *neglecting the terms of order  $O(h^5)$  in the final form of the energy.*

## 6 A comparison with the general 6-parameter shell model

In this section we give an overview of the quantities appearing in the general 6-parameter shell model presented in [36]. Eremeyev and Pietraskiewicz have considered the classical multiplicative decomposition  $F = F^e F^0$  of the (reconstructed) total deformation gradient into elastic and initial (“plastic”) parts [62, 57], i.e.  $F^e$  represents the (reconstructed) elastic shell deformation gradient, while  $F^0 = P$  is the initial deformation gradient. In the

general 6-parameter shell model, the following form for the elastic strain tensor  $E^e$  is used (written in matrix notation)

$$E^e = \overline{Q}_e^T (\partial_1 m | \partial_2 m | \overline{Q}_e n_0) P^{-1} - \mathbb{1}_3, \quad \text{or} \quad E^e = \overline{U}^e - \mathbb{1}_3 = \overline{Q}_e^T F^e - \mathbb{1}_3 = \overline{Q}_e^T F P^{-1} - \mathbb{1}_3, \quad (6.1)$$

with

$$\begin{aligned} F &= F^e F^0, & F &:= (\partial_1 m | \partial_2 m | \overline{Q}_e n_0) = (\nabla m | \overline{Q}_e n_0), \\ F^e &:= (\partial_1 m | \partial_2 m | \overline{Q}_e n_0) P^{-1}, & F^0 &:= P = (\partial_1 y_0 | \partial_2 y_0 | n_0) = (\nabla y_0 | n_0), \\ \overline{U}^e &= \overline{Q}_e^T (\partial_1 m | \partial_2 m | Q^e n_0) P^{-1} = \overline{Q}_e^T (\nabla m | \overline{Q}_e n_0) P^{-1}. \end{aligned} \quad (6.2)$$

Since,  $\nabla_x \Theta(0) = (\nabla y_0 | n_0) = P$  and  $n_0 = Q_0 \cdot e_3 = [\nabla_x \Theta(0)] \cdot e_3$ , we remark that  $E^e = \mathcal{E}_{m,s}$ . Hence, in the general 6-parameter shell model the same elastic shell strain tensor  $\mathcal{E}_{m,s}$  as in our shell model is used.

Regarding the bending curvature tensor, in the general 6-parameter shell model the tensor  $\mathcal{K}$  is the total bending–curvature tensor, while  $\mathcal{K}^0$  is the initial bending-curvature (or structure curvature tensor of  $\Omega_h$ ). The matrix  $\mathcal{K}^e = \mathcal{K} - \mathcal{K}_0$  is given by

$$\mathcal{K}^e = (\text{axl}(\overline{Q}_e^T \partial_1 Q_e) | \text{axl}(\overline{Q}_e^T \partial_2 Q_e) | 0) P^{-1}, \quad (6.3)$$

or

$$\begin{aligned} \mathcal{K}^e &= Q_0 L P^{-1} = \mathcal{K} - \mathcal{K}^0 \quad \text{with} \\ L &:= (\text{axl}(R^T \partial_1 R) - \text{axl}(Q_0^T \partial_1 Q_0) | \text{axl}(R^T \partial_2 R) - \text{axl}(Q_0^T \partial_2 Q_0) | 0)_{3 \times 3}, \\ \mathcal{K} &= Q_0 (\text{axl}(R^T \partial_1 R) | \text{axl}(R^T \partial_2 R) | 0) P^{-1}, \quad \mathcal{K}^0 = Q_0 (\text{axl}(Q_0^T \partial_1 Q_0) | \text{axl}(Q_0^T \partial_2 Q_0) | 0) P^{-1}. \end{aligned} \quad (6.4)$$

Using again that  $P = \nabla_x \Theta(0)$ , we have that the total bending-curvature tensor from Pietraskiewicz and his collaborators coincides with the elastic shell-bending-curvature tensor from our model, i.e.  $\mathcal{K}^e = \mathcal{K}_{e,s}$ . Therefore, we conclude that:

**Remark 6.1.**

- 1) A direct comparison with our model shows that the strain tensors  $E^e$  and  $\mathcal{K}^e$  from the general 6-parameter shell model corresponds to the tensor  $\mathcal{E}_{m,s}$  and  $\mathcal{K}_{e,s}$ , respectively, from our model;
- 2) While the general 6-parameter shell model is not deduced from a three dimensional energy, the strain tensors  $E^e$  and  $\mathcal{K}^e$  are directly introduced in the model as work-conjugate strain measures [75], without any explanation about how a reconstructed (three-dimensional) ansatz, which minimizes (approximately) a three-dimensional variational problem, leads to the form of the constitutive tensors  $E^e$  and  $\mathcal{K}^e$ .
- 3) Contrary to the general 6-parameter shell model, in our description, the roles and the deduction of the strain tensors  $\mathcal{E}_{m,s}$  and  $\mathcal{K}_{e,s}$  is explained by the dimensional reduction method:
  - the elastic shell bending-curvature tensor  $\mathcal{K}_{e,s}$  is appearing in the model from the form of the three-dimensional curvature energy (see (2.2)), after using Nye’s formula (3.11), the chain rule (see (3.22)) and using the ansatz (4.1) (see (4.49)).
  - the elastic shell strain tensor  $\mathcal{E}_{m,s}$  is appearing in the modelling process after the ansatz for the (reconstructed) deformation gradient (4.32) is proposed and it is suggested by the expressions of  $\varrho_m^e$  and  $\varrho_b^e$  (see (4.46)), in order to satisfy (approximately) the Neumann plane-stress boundary conditions in the reference configuration.
  - while  $\varrho_m^e$  depends only on  $\mathcal{E}_{m,s}$ ,  $\varrho_b^e$  depends on both tensors  $\mathcal{E}_{m,s}$  and  $\mathcal{K}_{e,s}$ . This means that the symmetric thickness stretch about the midsurface is influenced only by the elastic shell strain tensor  $\mathcal{E}_{m,s}$ , while the asymmetric thickness stretch about the midsurface is influenced by both the elastic shell strain tensor  $\mathcal{E}_{m,s}$  and the elastic shell bending-curvature tensor  $\mathcal{K}_{e,s}$ .
- 4) The complete description and role of the involved tensors in the dimensional deduction process is the effect of three factors:
  - in the deduction of our model we start with a three-dimensional variational problem for an (three-dimensional) elastic body. A shell is actually a three-dimensional body.



- we use the matrix formulation in the entire modelling process.
- we propose a specific isotropic form for the three-dimensional curvature energy in the parent three-dimensional variational problem.

In the resultant 6-parameter theory of shells, the strain energy density for isotropic shells has been presented in various forms. The simplest expression  $W_P(\mathcal{E}_{m,s}, \mathcal{K}_{e,s})$  has been proposed in the papers [20, 21] in the form

$$2W_P(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) = C[\nu(\text{tr } \mathcal{E}_{m,s}^{\parallel})^2 + (1-\nu)\text{tr}((\mathcal{E}_{m,s}^{\parallel})^T \mathcal{E}_{m,s}^{\parallel})] + \alpha_s C(1-\nu) \|\mathcal{E}_{m,s}^T n_0\|^2 \\ + D[\nu(\text{tr } \mathcal{K}_{e,s}^{\parallel})^2 + (1-\nu)\text{tr}((\mathcal{K}_{e,s}^{\parallel})^T \mathcal{K}_{e,s}^{\parallel})] + \alpha_t D(1-\nu) \|\mathcal{K}_{e,s}^T n_0\|^2, \quad (6.5)$$

where the decompositions of  $\mathcal{E}_{m,s}$  and  $\mathcal{K}_{e,s}$  into two orthogonal directions (in the tangential plane and in the normal direction)<sup>7</sup> are considered

$$\mathcal{E}_{m,s}^{\parallel} = A_{y_0} \mathcal{E}_{m,s} = (\mathbb{1}_3 - n_0 \otimes n_0) \mathcal{E}_{m,s}, \quad \mathcal{K}_{e,s}^{\parallel} = A_{y_0} \mathcal{K}_{e,s} = (\mathbb{1}_3 - n_0 \otimes n_0) \mathcal{K}_{e,s}, \quad (6.6) \\ \mathcal{E}_{m,s}^{\perp} = (\mathbb{1}_3 - A_{y_0}) \mathcal{E}_{m,s} = n_0 \otimes n_0 \mathcal{E}_{m,s}, \quad \mathcal{K}_{e,s}^{\perp} = (\mathbb{1}_3 - A_{y_0}) \mathcal{K}_{e,s} = n_0 \otimes n_0 \mathcal{K}_{e,s}.$$

Here, we have used that, since  $A_{y_0} = \mathbb{1}_3 - (0|0|n_0)(0|0|n_0)^T = \mathbb{1}_3 - n_0 \otimes n_0$ , for all  $X \in \mathbb{R}^{3 \times 3}$  the following equalities holds

$$\|X^{\perp}\|^2 = \|(\mathbb{1}_3 - A_{y_0})X\|^2 = \langle X, (\mathbb{1}_3 - A_{y_0})^2 X \rangle = \langle X, (\mathbb{1}_3 - A_{y_0})X \rangle = \langle X, (0|0|n_0)(0|0|n_0)^T X \rangle \\ = \langle (0|0|n_0)^T X, (0|0|n_0)^T X \rangle = \|X(0|0|n_0)^T\|^2 = \|X^T(0|0|n_0)\|^2 = \|X^T n_0\|^2. \quad (6.7)$$

The constitutive coefficient  $C = \frac{Eh}{1-\nu^2}$  is the stretching (in-plane) stiffness of the shell,  $D = \frac{Eh^3}{12(1-\nu^2)}$  is the bending stiffness, and  $\alpha_s, \alpha_t$  are two shear correction factors. Also,  $E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$  and  $\nu = \frac{\lambda}{2(\lambda+\mu)}$  denote the Young modulus and Poisson ratio of the isotropic and homogeneous material. In the numerical treatment of non-linear shell problems, the values of the shear correction factors have been set to  $\alpha_s = 5/6$ ,  $\alpha_t = 7/10$  in [21]. The value  $\alpha_s = 5/6$  is a classical suggestion, which has been previously deduced analytically by Reissner in the case of plates [78, 55]. Also, the value  $\alpha_t = 7/10$  was proposed earlier in [73, see p.78] and has been suggested in the work [72]. However, the discussion concerning the possible values of shear correction factors for shells is long and controversial in the literature [55, 56].

We write the strain energy density (6.5) in the equivalent form

$$2W_P(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) = C(1-\nu) [\|\text{sym}(A_{y_0} \mathcal{E}_{m,s})\|^2 + \|\text{skew}(A_{y_0} \mathcal{E}_{m,s})\|^2] \\ + C\nu [\text{tr}(A_{y_0} \mathcal{E}_{m,s})]^2 + \alpha_s C(1-\nu) \|\mathcal{E}_{m,s}^T n_0\|^2 \\ + D(1-\nu) [\|\text{sym}(A_{y_0} \mathcal{K}_{e,s})\|^2 + \|\text{skew}(A_{y_0} \mathcal{K}_{e,s})\|^2] \\ + D\nu [\text{tr}(A_{y_0} \mathcal{K}_{e,s})]^2 + \alpha_t D(1-\nu) \|\mathcal{K}_{e,s}^T n_0\|^2. \quad (6.8)$$

The coefficients in (6.8) are expressed in terms of the Lamé constants of the material  $\lambda$  and  $\mu$  now by the relations

$$C\nu = \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} h, \quad C(1-\nu) = 2\mu h, \quad D\nu = \frac{\mu(\lambda+\mu)}{\lambda+2\mu} \frac{h^3}{3}, \quad D(1-\nu) = \mu \frac{h^3}{6}.$$

In [36], Eremeyev and Pietraszkiewicz have proposed a more general form of the strain energy density, namely

$$2W_{EP}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) = \alpha_1 (\text{tr } \mathcal{E}_{m,s}^{\parallel})^2 + \alpha_2 \text{tr}(\mathcal{E}_{m,s}^{\parallel})^2 + \alpha_3 \text{tr}((\mathcal{E}_{m,s}^{\parallel})^T \mathcal{E}_{m,s}^{\parallel}) + \alpha_4 \|\mathcal{E}_{m,s}^T n_0\|^2 \\ + \beta_1 (\text{tr } \mathcal{K}_{e,s}^{\parallel})^2 + \beta_2 \text{tr}(\mathcal{K}_{e,s}^{\parallel})^2 + \beta_3 \text{tr}((\mathcal{K}_{e,s}^{\parallel})^T \mathcal{K}_{e,s}^{\parallel}) + \beta_4 \|\mathcal{K}_{e,s}^T n_0\|^2. \quad (6.9)$$

Already, note the absence of coupling terms involving  $\mathcal{K}_{e,s}^{\parallel}$  and  $\mathcal{E}_{m,s}^{\parallel}$ . The eight coefficients  $\alpha_k, \beta_k$  ( $k = 1, 2, 3, 4$ ) can depend in general on the structure curvature tensor  $\mathcal{K}^0$  of the reference configuration. We can decompose

<sup>7</sup>Here we have used that  $\langle A_{y_0}, \mathbb{1}_3 - A_{y_0} \rangle = \langle A_{y_0}, \mathbb{1}_3 \rangle - \langle A_{y_0} A_{y_0}^T, \mathbb{1}_3 \rangle = \langle A_{y_0}, \mathbb{1}_3 \rangle - \langle A_{y_0}^2, \mathbb{1}_3 \rangle = \langle A_{y_0}, \mathbb{1}_3 \rangle - \langle A_{y_0}, \mathbb{1}_3 \rangle = 0$ .

the strain energy density (6.9) in the in-plane part  $W_{\text{plane-EP}}(\mathcal{E}_{m,s})$  and the curvature part  $W_{\text{curv-EP}}(\mathcal{K}_{e,s})$  and write their expressions in the form

$$\begin{aligned} W_{\text{EP}}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) &= W_{\text{plane-EP}}(\mathcal{E}_{m,s}) + W_{\text{curv-EP}}(\mathcal{K}_{e,s}), \\ 2W_{\text{plane-EP}}(\mathcal{E}_{m,s}) &= (\alpha_2 + \alpha_3) \|\text{sym } \mathcal{E}_{m,s}^{\parallel}\|^2 + (\alpha_3 - \alpha_2) \|\text{skew } \mathcal{E}_{m,s}^{\parallel}\|^2 + \alpha_1 (\text{tr}(\mathcal{E}_{m,s}^{\parallel}))^2 + \alpha_4 \|\mathcal{E}_{m,s}^T n^0\|^2, \\ 2W_{\text{curv-EP}}(\mathcal{K}_{e,s}) &= (\beta_2 + \beta_3) \|\text{sym } \mathcal{K}_{e,s}^{\parallel}\|^2 + (\beta_3 - \beta_2) \|\text{skew } \mathcal{K}_{e,s}^{\parallel}\|^2 + \beta_1 (\text{tr}(\mathcal{K}_{e,s}^{\parallel}))^2 + \beta_4 \|\mathcal{K}_{e,s}^T n^0\|^2. \end{aligned} \quad (6.10)$$

Since in all the energies presented until now in this section, there exists no coupling terms in  $\mathcal{E}_{m,s}$  and  $\mathcal{K}_{e,s}$ , in the rest of this section, we compare them with a particular form of the energy proposed in our new model, i.e.

$$W_{\text{our}}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) = \left(h + K \frac{h^3}{12}\right) W_{\text{shell}}(\mathcal{E}_{m,s}) + \left(h - K \frac{h^3}{12}\right) W_{\text{curv}}(\mathcal{K}_{e,s}), \quad (6.11)$$

where

$$\begin{aligned} W_{\text{shell}}(S) &= \mu \|\text{sym } S\|^2 + \mu_c \|\text{skew } S\|^2 + \frac{\lambda \mu}{\lambda + 2\mu} [\text{tr } S]^2, \\ W_{\text{curv}}(S) &= \mu L_c^2 (b_1 \|\text{dev sym } S\|^2 + b_2 \|\text{skew } S\|^2 + 4b_3 [\text{tr}(S)]^2). \end{aligned}$$

To this aim, we consider the decompositions (6.6) and an arbitrary matrix  $S = (*|*|0) [\nabla_x \Theta(0)]^{-1}$ . Since  $A_{y_0}^2 = A_{y_0} \in \text{Sym}(3)$  and  $SA_{y_0} = S$  we have

$$\langle (\mathbb{1}_3 - A_{y_0}) S, A_{y_0} S \rangle = \langle (A_{y_0} - A_{y_0}^2) S, S \rangle = 0,$$

but also

$$(\mathbb{1}_3 - A_{y_0}) S^T = (S(\mathbb{1}_3 - A_{y_0}))^T = (S - SA_{y_0})^T = 0, \quad (6.12)$$

and consequently

$$\langle S^T (\mathbb{1}_3 - A_{y_0}), A_{y_0} S \rangle = 0 \quad \text{as well as} \quad \langle S^T (\mathbb{1}_3 - A_{y_0}), (\mathbb{1}_3 - A_{y_0}) S \rangle = 0.$$

Hence, we deduce that for all  $S = (*|*|0) \cdot [\nabla_x \Theta(0)]^{-1}$  we have the following split in the expression of the considered quadratic forms

$$\begin{aligned} W_{\text{shell}}(S) &= \mu \|\text{sym } S^{\parallel}\|^2 + \mu_c \|\text{skew } S^{\parallel}\|^2 + \frac{\mu + \mu_c}{2} \|S^{\perp}\|^2 + \frac{\lambda \mu}{\lambda + 2\mu} [\text{tr}(S)]^2, \\ W_{\text{curv}}(S) &= \mu L_c^2 \left( b_1 \|\text{sym } S^{\parallel}\|^2 + b_2 \|\text{skew } S^{\parallel}\|^2 + \frac{b_1 + b_2}{2} \|S^{\perp}\|^2 + \frac{12b_3 - b_1}{3} [\text{tr}(S)]^2 \right), \end{aligned}$$

where we have set  $S^{\parallel} := A_{y_0} S$  and  $S^{\perp} := (\mathbb{1}_3 - A_{y_0}) S$ . Moreover, using that for all  $S = (*|*|0) [\nabla_x \Theta(0)]^{-1}$  it holds true that

$$\text{tr}(S^{\perp}) = \text{tr}((\mathbb{1}_3 - A_{y_0})S) = \text{tr}(S) - \text{tr}(A_{y_0}S) = \text{tr}(S) - \text{tr}(SA_{y_0}) = 0, \quad (6.13)$$

we obtain for our model

$$\begin{aligned} W_{\text{shell}}(\mathcal{E}_{m,s}) &= \mu \|\text{sym } \mathcal{E}_{m,s}^{\parallel}\|^2 + \mu_c \|\text{skew } \mathcal{E}_{m,s}^{\parallel}\|^2 + \frac{\lambda \mu}{\lambda + 2\mu} [\text{tr}(\mathcal{E}_{m,s}^{\parallel})]^2 + \frac{\mu + \mu_c}{2} \|\mathcal{E}_{m,s}^{\perp}\|^2 \\ &= \mu \|\text{sym } \mathcal{E}_{m,s}^{\parallel}\|^2 + \mu_c \|\text{skew } \mathcal{E}_{m,s}^{\parallel}\|^2 + \frac{\lambda \mu}{\lambda + 2\mu} [\text{tr}(\mathcal{E}_{m,s}^{\parallel})]^2 + \frac{\mu + \mu_c}{2} \|\mathcal{E}_{m,s}^T n_0\|^2, \end{aligned} \quad (6.14)$$

and

$$W_{\text{curv}}(\mathcal{K}_{e,s}) = \mu L_c^2 \left( b_1 \|\text{sym } \mathcal{K}_{e,s}^{\parallel}\|^2 + b_2 \|\text{skew } \mathcal{K}_{e,s}^{\parallel}\|^2 + \frac{12b_3 - b_1}{3} [\text{tr}(\mathcal{K}_{e,s}^{\parallel})]^2 + \frac{b_1 + b_2}{2} \|\mathcal{K}_{e,s}^{\perp}\|^2 \right). \quad (6.15)$$

For the final comparison between the models, we rewrite our particular energy in the form

$$\begin{aligned}
W_{\text{our}}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) &= \left(h + \mathbb{K} \frac{h^3}{12}\right) \left[ \mu \|\text{sym } \mathcal{E}_{m,s}^{\parallel}\|^2 + \mu_c \|\text{skew } \mathcal{E}_{m,s}^{\parallel}\|^2 + \frac{\lambda \mu}{\lambda + 2\mu} [\text{tr}(\mathcal{E}_{m,s}^{\parallel})]^2 + \frac{\mu + \mu_c}{2} \|\mathcal{E}_{m,s}^T n_0\|^2 \right] \\
&+ \left(h - \mathbb{K} \frac{h^3}{12}\right) \mu L_c^2 \left[ b_1 \|\text{sym } \mathcal{K}_{e,s}^{\parallel}\|^2 + b_2 \|\text{skew } \mathcal{K}_{e,s}^{\parallel}\|^2 + \frac{12b_3 - b_1}{3} [\text{tr}(\mathcal{K}_{e,s})]^2 \right. \\
&\quad \left. + \frac{b_1 + b_2}{2} \|\mathcal{K}_{e,s}^{\perp}\|^2 \right].
\end{aligned} \tag{6.16}$$

This allows us to conclude

**Remark 6.2.**

i) By comparing our  $W_{\text{our}}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s})$  with  $W_{\text{EP}}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s})$  we deduce the following identification of the constitutive coefficients  $\alpha_1, \dots, \alpha_4, \beta_1, \dots, \beta_4$

$$\begin{aligned}
\alpha_1 &= \left(h + \mathbb{K} \frac{h^3}{12}\right) \frac{2\mu\lambda}{2\mu + \lambda}, & \alpha_2 &= \left(h + \mathbb{K} \frac{h^3}{12}\right) (\mu - \mu_c), \\
\alpha_3 &= \left(h + \mathbb{K} \frac{h^3}{12}\right) (\mu + \mu_c), & \alpha_4 &= \left(h + \mathbb{K} \frac{h^3}{12}\right) (\mu + \mu_c), \\
\beta_1 &= 2 \left(h - \mathbb{K} \frac{h^3}{12}\right) \mu L_c^2 \frac{12b_3 - b_1}{3}, & \beta_2 &= \left(h - \mathbb{K} \frac{h^3}{12}\right) \mu L_c^2 (b_1 - b_2), \\
\beta_3 &= \left(h - \mathbb{K} \frac{h^3}{12}\right) \mu L_c^2 (b_1 + b_2), & \beta_4 &= \left(h - \mathbb{K} \frac{h^3}{12}\right) \mu L_c^2 (b_1 + b_2).
\end{aligned} \tag{6.17}$$

ii) We observe that

$$\mu_c^{\text{drill}} := \alpha_3 - \alpha_2 = 2 \left(h + \mathbb{K} \frac{h^3}{12}\right) \mu_c, \tag{6.18}$$

which means that the in-plane rotational couple modulus  $\mu_c^{\text{drill}}$  of the Cosserat shell model is determined by the Cosserat couple modulus  $\mu_c$  of the 3D Cosserat material.

iii) In our shell model, the constitutive coefficients are those from the three-dimensional formulation, while the influence of the curved initial shell configuration appears explicitly in the expression of the coefficients of the energies for the reduced two-dimensional variational problem.

iv) The major difference between our model and the previously considered general 6-parameter shell model is that we include terms up to order  $O(h^5)$  and that, even in the case of a simplified theory of order  $O(h^3)$ , additional mixed terms like the membrane-bending part  $W_{\text{memb,bend}}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s})$  and  $W_{\text{curv}}(\mathcal{K}_{e,s} \mathbf{B}_{y_0})$  are included, which are otherwise difficult to guess.

## 7 Final comments

In this article, using a step by step transparent method, we have extended the modelling from flat Cosserat shells (plate) to the most general case of initially curved isotropic Cosserat shells. For flat shells, in a numerical approach Sander et al. [82] have shown that the new shell model offers a very good concordance with available experiments in the framework of nonlinear shell modelling. Our ansatz allows for a consistent shell model up to order  $O(h^5)$  in the shell thickness. Interestingly, all  $O(h^5)$  terms in the shell energy depend on the initial curvature of the shell and vanish for a flat shell. The  $O(h^5)$  terms do not come up with a definite sign, such that the additional terms can be stabilizing as well as destabilizing, depending on the local shell geometry. However, all occurring material coefficients of the shell model are uniquely determined from the isotropic three-dimensional Cosserat model and the given initial geometry of the shell. Hence, in contrast to other Cosserat shell models, we give an explicit form of the curvature energy, and therefore, we fill a certain gap in the general 6-parameter shell theories, which all leave the precise structure of the constitutive equations wide open.

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## Appendix

### A.1 Notation

We denote by  $\mathbb{R}^{n \times n}$ ,  $n \in \mathbb{N}$ , the set of real  $n \times n$  second order tensors, written with capital letters. We adopt the usual abbreviations of Lie-group theory, i.e.,  $\text{GL}(n) = \{X \in \mathbb{R}^{n \times n} \mid \det(X) \neq 0\}$  the general linear group,  $\text{SL}(n) = \{X \in \text{GL}(n) \mid \det(X) = 1\}$ ,  $\text{O}(n) = \{X \in \text{GL}(n) \mid X^T X = \mathbb{1}_n\}$ ,  $\text{SO}(n) = \{X \in \text{GL}(n) \mid X^T X = \mathbb{1}_n, \det(X) = 1\}$  with corresponding Lie-algebras  $\mathfrak{so}(n) = \{X \in \mathbb{R}^{n \times n} \mid X^T = -X\}$  of skew symmetric tensors and  $\mathfrak{sl}(n) = \{X \in \mathbb{R}^{n \times n} \mid \text{tr}(X) = 0\}$  of traceless tensors. Here, for  $a, b \in \mathbb{R}^n$  we let  $\langle a, b \rangle_{\mathbb{R}^n}$  denote the scalar product on  $\mathbb{R}^n$  with associated (squared) vector norm  $\|a\|_{\mathbb{R}^n}^2 = \langle a, a \rangle_{\mathbb{R}^n}$ . The standard Euclidean scalar product on  $\mathbb{R}^{n \times n}$  is given by  $\langle X, Y \rangle_{\mathbb{R}^{n \times n}} = \text{tr}(XY^T)$ , and thus the (squared) Frobenius tensor norm is  $\|X\|^2 = \langle X, X \rangle_{\mathbb{R}^{n \times n}}$ . In the following we omit the index  $\mathbb{R}^n, \mathbb{R}^{n \times n}$ . The identity tensor on  $\mathbb{R}^{n \times n}$  will be denoted by  $\mathbb{1}_n$ , so that  $\text{tr}(X) = \langle X, \mathbb{1}_n \rangle$ . We let  $\text{Sym}(n)$  and  $\text{Sym}^+(n)$  denote the symmetric and positive definite symmetric tensors, respectively. For all  $X \in \mathbb{R}^{3 \times 3}$  we set  $\text{sym } X = \frac{1}{2}(X^T + X) \in \text{Sym}(3)$ , skew  $X = \frac{1}{2}(X - X^T) \in \mathfrak{so}(3)$  and the deviatoric part  $\text{dev } X = X - \frac{1}{n} \text{tr}(X) \mathbb{1}_n \in \mathfrak{sl}(n)$  and we have the *orthogonal Cartan-decomposition of the Lie-algebra*  $\mathfrak{gl}(3)$

$$\mathfrak{gl}(3) = \{\mathfrak{sl}(3) \cap \text{Sym}(3)\} \oplus \mathfrak{so}(3) \oplus \mathbb{R} \cdot \mathbb{1}_3, \quad X = \text{dev sym } X + \text{skew } X + \frac{1}{3} \text{tr}(X) \mathbb{1}_3. \quad (\text{A.1})$$

We make use of the operator  $\text{axl} : \mathfrak{so}(3) \rightarrow \mathbb{R}^3$  associating with a matrix  $A \in \mathfrak{so}(3)$  the vector  $\text{axl } A := (-A_{23}, A_{13}, -A_{12})^T$ .

For  $X \in \text{GL}(n)$ ,  $\text{Adj}(X)$  denotes the tensor of transposed cofactors, while the  $(i, j)$  entry of the cofactor is the  $(i, j)$ -minor times a sign factor. For vectors  $\xi, \eta \in \mathbb{R}^n$ , we have the tensor product  $(\xi \otimes \eta)_{ij} = \xi_i \eta_j$ . A matrix having the three column vectors  $A_1, A_2, A_3$

will be written as  $(A_1 \mid A_2 \mid A_3)$ . For a given matrix  $M \in \mathbb{R}^{2 \times 2}$  we define the lifted quantity  $M^b = \begin{pmatrix} M_{11} & M_{12} & 0 \\ M_{21} & M_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{3 \times 3}$ .

Let  $\Omega$  be an open domain of  $\mathbb{R}^3$ . The usual Lebesgue spaces of square integrable functions, vector or tensor fields on  $\Omega$  with values in  $\mathbb{R}$ ,  $\mathbb{R}^3$  or  $\mathbb{R}^{3 \times 3}$ , respectively will be denoted by  $L^2(\Omega)$ . Moreover, we introduce the standard Sobolev spaces  $H^1(\Omega) = \{u \in L^2(\Omega) \mid \nabla u \in L^2(\Omega)\}$ ,  $H(\text{curl}; \Omega) = \{v \in L^2(\Omega) \mid \text{curl } v \in L^2(\Omega)\}$  of functions  $u$  or vector fields  $v$ , respectively. For vector fields  $u = (u_1, u_2, u_3)$  with  $u_i \in H^1(\Omega)$ ,  $i = 1, 2, 3$ , we define

$$\nabla u := (\nabla u_1 \mid \nabla u_2 \mid \nabla u_3)^T,$$

while for tensor fields  $P$  with rows in  $H(\text{curl}; \Omega)$ , i.e.  $P = (P^T \cdot e_1 \mid P^T \cdot e_2 \mid P^T \cdot e_3)^T$  with  $(P^T \cdot e_i)^T \in H(\text{curl}; \Omega)$ ,  $i = 1, 2, 3$ , we define

$$\text{Curl } P := (\text{curl}(P^T \cdot e_1)^T \mid \text{curl}(P^T \cdot e_2)^T \mid \text{curl}(P^T \cdot e_3)^T)^T.$$

The corresponding Sobolev-spaces will be denoted by  $H^1(\Omega)$  and  $H^1(\text{Curl}; \Omega)$ , respectively. We will use the notations:  $\nabla_\xi$ ,  $\nabla_x$ ,  $\text{Curl}_\xi$ ,  $\text{Curl}_x$  etc. to indicate the variables for which these quantities are calculated.

## A.2 Prerequisites from classical differential geometry

Let  $\omega \subset \mathbb{R}^2$  be an open domain. A given regular mapping  $y_0 : \omega \rightarrow \mathbb{R}^3$ , describing a *surface imbedded in the three-dimensional space* is called *regular* whenever  $\text{rank}(\nabla y_0) = 2$ . The column vector

$$n_0 := \frac{\partial_{x_1} y_0 \times \partial_{x_2} y_0}{\|\partial_{x_1} y_0 \times \partial_{x_2} y_0\|} \quad (\text{A.2})$$

is the *Gauß unit normal field* on the surface. We need to compute

$$\text{Adj}[(\nabla y_0|0)] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \left| \begin{array}{cc} \partial_{x_1} y_{02} & \partial_{x_2} y_{02} \\ \partial_{x_1} y_{03} & \partial_{x_2} y_{03} \end{array} \right| & - \left| \begin{array}{cc} \partial_{x_1} y_{01} & \partial_{x_2} y_{01} \\ \partial_{x_1} y_{03} & \partial_{x_2} y_{03} \end{array} \right| & \left| \begin{array}{cc} \partial_{x_1} y_{01} & \partial_{x_2} y_{01} \\ \partial_{x_1} y_{02} & \partial_{x_2} y_{02} \end{array} \right| \end{pmatrix}. \quad (\text{A.3})$$

Hence, it follows

$$n_0 := \frac{\text{Cof}(\nabla y_0|0) \cdot e_3}{\|\text{Cof}(\nabla y_0|0) \cdot e_3\|}, \quad \text{Cof}(X) = [\text{Adj}(X)]^T \quad \forall X \in \mathbb{R}^{3 \times 3}. \quad (\text{A.4})$$

The map  $n_0 : \omega \rightarrow S^2$  is called the *Gauß map* (where  $S^2$  is the unit sphere in  $\mathbb{R}^3$ ) and the moving 3-frame  $(\partial_{x_1} y_0 | \partial_{x_2} y_0 | n_0)$  is called the *Gauß frame* of the surface  $y_0(\omega)$ , which in general is not orthonormal. The matrix representation of the *first fundamental form* (*metric*) on  $y_0(\omega)$  is given through

$$I_{y_0} := [\nabla y_0]^T \nabla y_0 = \begin{pmatrix} \|\partial_{x_1} y_0\|^2 & \langle \partial_{x_1} y_0, \partial_{x_2} y_0 \rangle \\ \langle \partial_{x_1} y_0, \partial_{x_2} y_0 \rangle & \|\partial_{x_2} y_0\|^2 \end{pmatrix} \in \mathbb{R}^{2 \times 2}. \quad (\text{A.5})$$

Because  $\text{rank}(\nabla y_0) = 2$ , the tensor  $[\nabla y_0]^T \nabla y_0$  is positive definite.

The metric alone is not sufficient to describe the shape of a surface in the ambient three-dimensional Euclidean space, the curvature is also needed. However, in the case  $(\nabla y_0|n_0) \in \text{SO}(3)$ , the metric is indeed enough. With the metric, the length and angles (and changes of length and angles) of a surface can be completely described.

The matrix representation of the *second fundamental form* on  $y_0(\omega)$  providing a measure for curvature of the surface is given by

$$II_{y_0} := -[\nabla y_0]^T \nabla n_0 = -(\partial_{x_1} y_0 | \partial_{x_2} y_0)^T (\partial_{x_1} n_0 | \partial_{x_2} n_0) = - \left( \left\langle \begin{array}{c} \partial_{x_1} y_0, \partial_{x_1} n_0 \\ \partial_{x_2} y_0, \partial_{x_1} n_0 \end{array} \right\rangle \left\langle \begin{array}{c} \partial_{x_1} y_0, \partial_{x_2} n_0 \\ \partial_{x_2} y_0, \partial_{x_2} n_0 \end{array} \right\rangle \right) \in \mathbb{R}^{2 \times 2}. \quad (\text{A.6})$$

Since  $n_0$  is orthogonal to the tangent space  $T_x y_0$  of the surface  $y_0$ , the relation  $0 = \partial_{x_1} \langle \partial_{x_2} y_0, n_0 \rangle = \partial_{x_2} \langle \partial_{x_1} y_0, n_0 \rangle$  shows easily that  $II_{y_0} \in \text{Sym}(2)$ .

The *third fundamental form* of the surface  $y_0(\omega)$  in matrix representation is defined as

$$III_{y_0} := \nabla n_0^T \nabla n_0 = \begin{pmatrix} \|\partial_{x_1} n_0\|^2 & \langle \partial_{x_1} n_0, \partial_{x_2} n_0 \rangle \\ \langle \partial_{x_2} n_0, \partial_{x_1} n_0 \rangle & \|\partial_{x_2} n_0\|^2 \end{pmatrix} \in \mathbb{R}^{2 \times 2}. \quad (\text{A.7})$$

Since  $[\nabla y_0]^T \nabla y_0$  is positive definite, the first fundamental form induces a scalar product  $g(\xi_1, \xi_2) := \langle I_{y_0} \xi_1, \xi_2 \rangle_{\mathbb{R}^2}$ , while the second fundamental form induces a symmetric bilinear form  $\tilde{g}(\xi_1, \xi_2) := \langle II_{y_0} \xi_1, \xi_2 \rangle_{\mathbb{R}^2}$ .

The first fundamental form and the second fundamental form are connected by *the Weingarten map* (or shape operator) which we again identify with the associated matrix  $L_{y_0} \in \mathbb{R}^{2 \times 2}$ , i.e.  $\forall \xi_1, \xi_2 \in \mathbb{R}^2$  we have  $\langle I_{y_0} \xi_1, L_{y_0} \xi_2 \rangle_{\mathbb{R}^2} = \langle II_{y_0} \xi_1, \xi_2 \rangle_{\mathbb{R}^2}$ . This is an implicit definition of the Weingarten map  $L_{y_0}$ . Using the definitions of the first and the second fundamental form, we have

$$\begin{aligned} \langle [\nabla y_0]^T \nabla y_0 \xi_1, L_{y_0} \xi_2 \rangle_{\mathbb{R}^2} &= \langle -[\nabla y_0]^T \nabla n_0 \xi_1, \xi_2 \rangle_{\mathbb{R}^2} \\ \Leftrightarrow \langle (L_{y_0}^T [\nabla y_0]^T \nabla y_0 + [\nabla y_0]^T \nabla n_0) \xi_1, \xi_2 \rangle_{\mathbb{R}^2} &= 0, \quad \forall \xi_1, \xi_2 \in \mathbb{R}^2. \end{aligned} \quad (\text{A.8})$$

Thus, we have

$$L_{y_0}^T [\nabla y_0]^T \nabla y_0 + [\nabla y_0]^T \nabla n_0 = 0 \quad \Leftrightarrow \quad L_{y_0}^T [\nabla y_0]^T \nabla y_0 = -[\nabla y_0]^T \nabla n_0. \quad (\text{A.9})$$

Hence, we obtain the following alternative expression for the Weingarten map via the so called *Weingarten equations* :

$$L_{y_0} = -([\nabla y_0]^T \nabla y_0)^{-1} (\nabla n_0^T \nabla y_0) \quad \text{or} \quad L_{y_0} = I_{y_0}^{-1} II_{y_0}. \quad (\text{A.10})$$

Moreover, using the symmetry of the second fundamental form we see that the Weingarten map satisfies:

$$\nabla y_0 L_{y_0} = -\nabla n_0. \quad (\text{A.11})$$

We have also

$$III_{y_0} = I_{y_0} L_{y_0}^2 = II_{y_0} I_{y_0}^{-1} II_{y_0}. \quad (\text{A.12})$$

The *Gauß curvature*  $K$  of the surface  $y_0(\omega)$  is determined by

$$K := \det(II_{y_0} I_{y_0}^{-1}) = \det(L_{y_0}), \quad (\text{A.13})$$

and the *mean curvature*  $H$  through

$$2H := \text{tr}(L_{y_0}). \quad (\text{A.14})$$



The following classification is standard. The surface  $y_0$  is locally

$$\begin{cases} \text{elliptic} \\ \text{parabolic} \\ \text{hyperbolic} \\ \text{planar} \end{cases} \quad \text{at } (x_1, x_2) \in \omega \quad \text{if } K \text{ is } \begin{cases} > 0 \\ = 0 \text{ and } H \neq 0 \\ < 0 \\ = 0 \text{ and } H = 0 \end{cases}. \quad (\text{A.15})$$

It is well known that  $H = 0$  is satisfied for all sufficiently regular stationary points of the minimal surface area functional. The Caley-Hamilton theorem implies

$$L_{y_0}^2 - 2HL_{y_0} + K\mathbb{1}_2 = 0.$$

Thus, the relation

$$III_{y_0} - 2HII_{y_0} + KI_{y_0} = 0$$

([48, Prop. 3.5.6]) is a consequence of the Caley-Hamilton theorem and shows that  $III_{y_0}$  is symmetric and is not independent of  $I_{y_0}, II_{y_0}$ . The principal curvatures  $\kappa_1, \kappa_2$  are the solutions of the characteristic equation of  $L_{y_0}$ , i.e.

$$\kappa^2 - \text{tr}(L_{y_0})\kappa + \det(L_{y_0}) = \kappa^2 - 2H\kappa + K = 0.$$

We define the lifted quantity  $\widehat{I}_{y_0} \in \mathbb{R}^{3 \times 3}$  by

$$\widehat{I}_{y_0} = (\nabla y_0 | n_0)^T (\nabla y_0 | n_0) = \begin{pmatrix} \|\partial_{x_1} y_0\|^2 & \langle \partial_{x_1} y_0, \partial_{x_2} y_0 \rangle & 0 \\ \langle \partial_{x_1} y_0, \partial_{x_2} y_0 \rangle & \|\partial_{x_2} y_0\|^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} I_{y_0} & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_{y_0}^b + \widehat{0}_3, \quad (\text{A.16})$$

where  $I_{y_0}^b = \begin{pmatrix} I_{y_0} & 0 \\ 0 & 0 & 0 \end{pmatrix}$  and  $\widehat{0}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ . Hence,  $\widehat{I}_{y_0}$  has the properties

$$\det(I_{y_0}) = \det(\widehat{I}_{y_0}) = \det(\nabla y_0 | n_0)^2, \quad \text{tr}(I_{y_0}) + 1 = \text{tr}(\widehat{I}_{y_0}).$$

Corresponding to the second fundamental form we define the lifted quantity  $\widehat{\Pi}_{y_0} \in \mathbb{R}^{3 \times 3}$  by

$$\widehat{\Pi}_{y_0} = -(\nabla y_0 | n_0)^T (\nabla n_0 | n_0) = - \begin{pmatrix} \langle \partial_{x_1} y_0, \partial_{x_1} n_0 \rangle & \langle \partial_{x_1} y_0, \partial_{x_2} n_0 \rangle & 0 \\ \langle \partial_{x_2} y_0, \partial_{x_1} n_0 \rangle & \langle \partial_{x_2} y_0, \partial_{x_2} n_0 \rangle & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} II_{y_0} & 0 \\ 0 & 0 & -1 \end{pmatrix} = II_{y_0}^b - \widehat{0}_3, \quad (\text{A.17})$$

where  $II_{y_0}^b = \begin{pmatrix} II_{y_0} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ . It has the properties

$$\det(II_{y_0}) = -\det(\widehat{\Pi}_{y_0}), \quad \text{tr}(II_{y_0}) = \text{tr}(\widehat{\Pi}_{y_0}) + 1. \quad (\text{A.18})$$

Let us consider as well the lifted Weingarten map  $\widehat{L}_{y_0} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  defined by

$$\widehat{L}_{y_0}^T = \widehat{\Pi}_{y_0} \widehat{I}_{y_0}^{-1}. \quad (\text{A.19})$$

Thus, we have

$$\begin{aligned} \widehat{L}_{y_0}^T &= -(\nabla y_0 | n_0)^T (\nabla n_0 | n_0) [(\nabla y_0 | n_0)^T (\nabla y_0 | n_0)]^{-1} \\ &= \begin{pmatrix} II_{y_0} & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} I_{y_0}^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} II_{y_0} I_{y_0}^{-1} & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} L_{y_0}^T & 0 \\ 0 & 0 & -1 \end{pmatrix}. \end{aligned} \quad (\text{A.20})$$

The lifted Weingarten map  $\widehat{L}$  has the following properties

$$\det(\widehat{L}_{y_0}) = -\det(L_{y_0}), \quad \text{tr}(\widehat{L}_{y_0}) = \text{tr}(L_{y_0}) - 1. \quad (\text{A.21})$$

### A.3 Properties of the diffeomorphism $\Theta$

**Lemma A.1.** *For all  $A \in \mathbb{R}^{2 \times 2}$ , there exists  $a > 0$ , such that for all  $x_3 \in (-a, a)$ , the formula*

$$(\mathbb{1}_2 - x_3 A)^{-1} = \frac{1}{1 - x_3 \text{tr} A + x_3^2 \det A} [(1 - x_3 \text{tr} A) \mathbb{1}_2 + x_3 A] \quad (\text{A.22})$$

holds true.

*Proof.* Consider an arbitrary matrix  $A \in \mathbb{R}^{2 \times 2}$ . From the continuity of the mapping  $x_3 \mapsto 1 - x_3 \text{tr}(A) + x_3^2 \det(A)$ , it exists an  $a > 0$  such that  $1 - x_3 \text{tr}(A) + x_3^2 \det(A) > 0$  for all  $x_3 \in (-a, a)$ . Taking  $x_3 \in (-a, a)$ , we compute

$$\begin{aligned} (\mathbb{1}_2 - x_3 A)^{-1} (\mathbb{1}_2 - x_3 A) &= \frac{1}{1 - x_3 \text{tr}(A) + x_3^2 \det(A)} [(1 - x_3 \text{tr}(A)) \mathbb{1}_2 + x_3 A] [\mathbb{1}_2 - x_3 A] \\ &= \frac{1}{1 - x_3 \text{tr}(A) + x_3^2 \det(A)} [(1 - x_3 \text{tr}(A)) \mathbb{1}_2 - x_3 A + x_3^2 \text{tr}(A) A + x_3 A - x_3^2 A^2] \\ &= \frac{1}{1 - x_3 \text{tr} A + x_3^2 \det A} [(1 - x_3 \text{tr}(A)) \mathbb{1}_2 + x_3^2 \underbrace{(\text{tr}(A) A - A^2)}_{\det(A) \mathbb{1}_2}] = \mathbb{1}_2. \quad \blacksquare \end{aligned} \quad (\text{A.23})$$

With the help of the above lemma, we prove the following proposition.

**Proposition A.2.** *The diffeomorphism  $\Theta$  has the following properties for all  $x_3$ :*

- i)  $\det(\nabla_x \Theta(x_3)) = \det(\nabla_{y_0}|n_0) [1 - 2x_3 H + x_3^2 K]$ ;
- ii)  $\nabla_x \Theta(x_3)$  belongs to  $\text{GKC} := \{X \in \text{GL}^+(3) \mid X^T X.e_3 = \varrho^2 e_3, \varrho \in \mathbb{R}^+\}$ ;
- iii) if  $h|\kappa_1| < \frac{1}{2}, h|\kappa_2| < \frac{1}{2}$ , then for all  $x_3 \in \left(-\frac{h}{2}, \frac{h}{2}\right)$ :

$$[\nabla_x \Theta(x_3)]^{-1} = \frac{1}{1 - 2Hx_3 + Kx_3^2} \left[ \mathbb{1}_3 + x_3(L_{y_0}^b - 2H\mathbb{1}_3) + x_3^2 K \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] [\nabla_x \Theta(0)]^{-1};$$

*Proof.* Since  $\|n_0\|^2 = 1$  we have  $\langle n_0, \partial_{x_\alpha} n_0 \rangle = 0, \langle n_0, \partial_{x_\alpha} y_0 \rangle = 0, \alpha = 1, 2$ . Using the Weingarten map (or shape operator)  $L_{y_0} \in \mathbb{R}^{2 \times 2}$  defined in Appendix A.2 by relation (A.11) we deduce the following form of  $\nabla_x \Theta$ :

$$\nabla_x \Theta(x_3) = (\nabla_{y_0}|n_0) + x_3(\nabla n_0|0) = (\nabla_{y_0}|n_0) - x_3(\nabla_{y_0} L_{y_0}|0), \quad (\text{A.24})$$

which implies

$$\nabla_x \Theta(x_3) = \nabla_x \Theta(0) \begin{pmatrix} \mathbb{1}_2 - x_3 L_{y_0} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (\text{A.25})$$

Then, we have  $\det(\nabla_x \Theta(x_3)) = \det(\nabla_x \Theta(0)) \det(\mathbb{1}_2 - x_3 L_{y_0})$ . Using the two-dimensional expansion of the determinant  $\det(\mathbb{1}_2 - x_3 L_{y_0}) = 1 - x_3 \text{tr}(L_{y_0}) + x_3^2 \det(L_{y_0})$ , we deduce  $\det(\nabla_x \Theta(x_3)) = \det(\nabla_x \Theta(0)) [1 - x_3 \text{tr}(L_{y_0}) + x_3^2 \det(L_{y_0})]$ . In terms of the mean curvature  $H$  and the Gauss curvature  $K$ , we have therefore the well known formula i).

Regarding ii), we have already seen in (A.16) that  $\nabla_x \Theta(0) = (\nabla_{y_0}|n_0) \in \text{GKC}$ , so that the conclusion  $\nabla_x \Theta(x_3) \in \text{GKC}$  then follows from the decomposition (A.25).

In order to prove iii), we prove that the conditions  $h|\kappa_1| < \frac{1}{2}, h|\kappa_2| < \frac{1}{2}$ , ensure that

$$1 - 2Hx_3 + Kx_3^2 \neq 0 \quad \text{for all} \quad x_3 \in \left(-\frac{h}{2}, \frac{h}{2}\right).$$

It follows that  $h^2|K| = h^2|\kappa_1||\kappa_2| < \frac{1}{4}, 2h|H| = h|\kappa_1 + \kappa_2| < 1$ . Hence,  $1 - 2Hx_3 + Kx_3^2 \geq 1 - 2|H||x_3| - |K||x_3|^2 > 0$ . Thus, we use the representation (A.25) and we apply Lemma A.1 and iii) is proven.  $\blacksquare$

## A.4 Properties of the tensors $A_{y_0}, B_{y_0}$ and $C_{y_0}$

**Proposition A.3.** *The tensors  $A_{y_0}, B_{y_0}$  and  $C_{y_0}$  have the following properties:*

- i)  $A_{y_0} = [\nabla_x \Theta(0)]^{-T} I_{y_0}^b [\nabla_x \Theta(0)]^{-1} \in \text{Sym}(3), \quad \text{tr}(A_{y_0}) = 2, \quad \det(A_{y_0}) = 0,$   
 $A_{y_0} = \mathbb{1}_3 - (0|0|\nabla_x \Theta(0).e_3) [\nabla_x \Theta(0)]^{-1} = \mathbb{1}_3 - (0|0|n_0)(0|0|n_0)^T;$
- ii)  $B_{y_0} = [\nabla_x \Theta(0)]^{-T} \Pi_{y_0}^b [\nabla_x \Theta(0)]^{-1} = \nabla_x \Theta(0) L_{y_0}^b [\nabla_x \Theta(0)]^{-1} \in \text{Sym}(3),$   
 $\text{tr}(B_{y_0}) = 2H, \quad \det[B_{y_0}] = 0, \quad \text{tr}(\text{Cof } B_{y_0}) = K, \quad \text{Cof } B_{y_0} = \nabla_x \Theta(0) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & K \end{pmatrix} [\nabla_x \Theta(0)]^{-1},$   
 $B_{y_0}^2 = [\nabla_x \Theta(0)]^{-T} \text{III}_{y_0}^b [\nabla_x \Theta(0)]^{-1} = [\nabla_x \Theta(0)] (L_{y_0}^b)^2 [\nabla_x \Theta(0)]^{-1};$
- iii)  $B_{y_0}$  satisfies the equation of Cayley-Hamilton type  $B_{y_0}^2 - 2HB_{y_0} + KA_{y_0} = 0_3;$
- iv)  $A_{y_0} B_{y_0} = B_{y_0} A_{y_0} = B_{y_0}, \quad A_{y_0}^2 = A_{y_0};$
- v)  $(u_1|u_2|0) [\nabla_x \Theta(0)]^{-1} A_{y_0} = (u_1|u_2|0) [\nabla_x \Theta(0)]^{-1}$  for all  $u_1, u_2 \in \mathbb{R}^3;$
- vi)  $C_{y_0} \in \mathfrak{so}(3), C_{y_0}^2 = -A_{y_0}$  and it has the simplified form  $C_{y_0} := Q_0(0) \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q_0^T(0);$
- vii)  $B_{y_0} = -C_{y_0} Q_0(0) \left( \text{axl}(Q_0^T(0) \partial_{x_1} Q_0(0)) \mid \text{axl}(Q_0^T(0) \partial_{x_2} Q_0(0)) \mid 0 \right) [\nabla_x \Theta(0)]^{-1}.$

*Proof.* i) We deduce

$$\begin{aligned} A_{y_0} &= (\nabla_{y_0}|0) [\nabla_x \Theta(0)]^{-1} = (\nabla_{y_0}|n_0) \mathbb{1}_2^b [\nabla_x \Theta(0)]^{-1} = [\nabla_x \Theta(0)]^{-T} [\nabla_x \Theta(0)]^T [\nabla_x \Theta(0)] \mathbb{1}_2^b [\nabla_x \Theta(0)]^{-1} \\ &= [\nabla_x \Theta(0)]^{-T} \hat{I}_{y_0} \mathbb{1}_2^b [\nabla_x \Theta(0)]^{-1} = [\nabla_x \Theta(0)]^{-T} I_{y_0}^b [\nabla_x \Theta(0)]^{-1}. \end{aligned} \quad (\text{A.26})$$

Therefore, the first identity of i) is proven and it also follows that

$$\text{tr}(A_{y_0}) = \langle [\nabla_x \Theta(0)]^{-T} I_{y_0}^b [\nabla_x \Theta(0)]^{-1}, \mathbb{1}_3 \rangle = \langle I_{y_0}^b, [\nabla_x \Theta(0)]^{-1} [\nabla_x \Theta(0)]^{-T} \rangle = \langle I_{y_0}^b, (I_{y_0}^{-1})^b \rangle = 2. \quad (\text{A.27})$$

We use that  $\nabla_x \Theta(x_3) \in \text{GKC}$  and calculate

$$\begin{aligned}
(\nabla_{y_0}|0)(\nabla_{y_0}|n_0)^{-1} &= (\nabla_{y_0}|n_0)(\nabla_{y_0}|n_0)^{-1} - (0|0|n_0)(\nabla_{y_0}|n_0)^{-1} \\
&= \mathbb{1}_3 - (0|0|n_0)(\nabla_{y_0}|n_0)^{-1} = \mathbb{1}_3 - (0|0|n_0)U_0^{-1}(0)Q_0^T(0) \\
&= \mathbb{1}_3 - (0|0|n_0) \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix} Q_0^T(0) = \mathbb{1}_3 - (0|0|n_0)Q_0^T(0) \\
&= \mathbb{1}_3 - (0|0|n_0)(d_1^0(0)|d_2^0(0)|n_0)^T = \mathbb{1}_3 - (0|0|n_0)(0|0|n_0)^T.
\end{aligned} \tag{A.28}$$

The last identity of i) follows directly from (3.29).

ii) In terms of the second fundamental form the tensor  $B_{y_0}$  has the form

$$B_{y_0} = -[\nabla_x \Theta(0)]^{-T} (\nabla_{y_0}|n_0)^T (\nabla_{y_0}|0) [\nabla_x \Theta(0)]^{-1} = [\nabla_x \Theta(0)]^{-T} \Pi_{y_0}^b [\nabla_x \Theta(0)]^{-1}, \tag{A.29}$$

from which symmetry follows, since  $\Pi_{y_0}^b$  is symmetric. Moreover, we have

$$[\nabla_x \Theta(0)]^{-1} B_{y_0} = [\nabla_x \Theta(0)]^{-1} [\nabla_x \Theta(0)]^{-T} \Pi_{y_0}^b [\nabla_x \Theta(0)]^{-1} = \begin{pmatrix} I_{y_0}^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Pi_{y_0} & 0 \\ 0 & 0 & 0 \end{pmatrix} [\nabla_x \Theta(0)]^{-1} = L_{y_0}^b [\nabla_x \Theta(0)]^{-1}. \tag{A.30}$$

The identity  $\det[B_{y_0}] = 0$  follows directly from (3.29). A direct consequence of the above relation is

$$\text{tr}(B_{y_0}) = \text{tr}([\nabla_x \Theta(0)] L_{y_0}^b [\nabla_x \Theta(0)]^{-1}) = \text{tr}(L_{y_0}^b) = \text{tr}(L_{y_0}) = 2H. \tag{A.31}$$

To compute  $\text{Cof } B_{y_0}$  we use that  $\text{Cof}(XY) = \text{Cof}(X) \text{Cof}(Y)$  for any  $X, Y \in \mathbb{R}^{3 \times 3}$ . It follows

$$\begin{aligned}
\text{Cof } B_{y_0} &= \text{Cof} [\nabla_x \Theta(0)] \text{Cof} (L_{y_0}^b) \text{Cof} [\nabla_x \Theta(0)]^{-1} \\
&= \det [\nabla_x \Theta(0)] \cdot [\nabla_x \Theta(0)]^{-T} \text{Cof} \begin{pmatrix} L_{y_0} & 0 \\ 0 & 0 & 0 \end{pmatrix} [\nabla_x \Theta(0)]^T \det [\nabla_x \Theta(0)]^{-1} \\
&= [\nabla_x \Theta(0)]^{-T} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & K \end{pmatrix} [\nabla_x \Theta(0)]^T = (0|0|n_0) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & K \end{pmatrix} (0|0|n_0)^T = [\nabla_x \Theta(0)] \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & K \end{pmatrix} [\nabla_x \Theta(0)]^{-1}.
\end{aligned} \tag{A.32}$$

The relation for  $B_{y_0}^2$  can be proved similarly without difficulties.

iii) In the following we prove the Cayley-Hamilton type equation. Using ii), we deduce

$$\begin{aligned}
B_{y_0}^2 - 2H B_{y_0} + K A_{y_0} &= \nabla_x \Theta(0) (L_{y_0}^b)^2 [\nabla_x \Theta(0)]^{-1} - 2H \nabla_x \Theta(0) L_{y_0}^b [\nabla_x \Theta(0)]^{-1} + K (\nabla_{y_0}|n_0) \mathbb{1}_2^b [\nabla_x \Theta(0)]^{-1} \\
&= \nabla_x \Theta(0) (L_{y_0}^2 - 2H L_{y_0} + K \mathbb{1}_2)^b [\nabla_x \Theta(0)]^{-1} = 0_3.
\end{aligned} \tag{A.33}$$

iv) In order to prove iv), we deduce from i) and ii)

$$\begin{aligned}
A_{y_0} B_{y_0} &= [\nabla_x \Theta(0)]^{-T} \Pi_{y_0}^b [\nabla_x \Theta(0)]^{-1} \nabla_x \Theta(0) L_{y_0}^b [\nabla_x \Theta(0)]^{-1} \\
&= [\nabla_x \Theta(0)]^{-T} \Pi_{y_0}^b L_{y_0}^b [\nabla_x \Theta(0)]^{-1} = [\nabla_x \Theta(0)]^{-T} \Pi_{y_0}^b [\nabla_x \Theta(0)]^{-1} = B_{y_0}.
\end{aligned} \tag{A.34}$$

Moreover, we have

$$B_{y_0} A_{y_0} = B_{y_0} (\mathbb{1}_3 - (0|0|n_0)(0|0|n_0)^T) = B_{y_0} - \nabla_x \Theta(0) L_{y_0}^b [\nabla_x \Theta(0)]^{-1} (0|0|n_0)(0|0|n_0)^T. \tag{A.35}$$

We notice that

$$\begin{aligned}
[\nabla_x \Theta(0)]^{-1} (0|0|n_0) &= (0|0|e_3) \Rightarrow [\nabla_x \Theta(0)]^{-1} (0|0|n_0)(0|0|n_0)^T = (0|0|e_3)(0|0|n_0)^T = (0|0|n_0)^T \\
&\Rightarrow L_{y_0}^b [\nabla_x \Theta(0)]^{-1} (0|0|n_0)(0|0|n_0)^T = L_{y_0}^b (0|0|n_0)^T = (0|0|0) = 0_3.
\end{aligned} \tag{A.36}$$

Using (A.36) in (A.35) we obtain  $B_{y_0} A_{y_0} = B_{y_0}$ . Similarly, from (A.36) we find

$$\begin{aligned}
A_{y_0}^2 &= A_{y_0} (\mathbb{1}_3 - (0|0|n_0)(0|0|n_0)^T) = A_{y_0} - [\nabla_x \Theta(0)]^{-T} \Pi_{y_0}^b [\nabla_x \Theta(0)]^{-1} (0|0|n_0)(0|0|n_0)^T \\
&= A_{y_0} - [\nabla_x \Theta(0)]^{-T} \Pi_{y_0}^b (0|0|n_0)^T = A_{y_0}.
\end{aligned}$$

v) We consider two vectors  $u_1, u_2 \in \mathbb{R}^3$  and we compute

$$\begin{aligned}
(u_1|u_2|0) [\nabla_x \Theta(0)]^{-1} A_{y_0} &= (u_1|u_2|0) [\nabla_x \Theta(0)]^{-1} - (u_1|u_2|0) [\nabla_x \Theta(0)]^{-1} (0|0|\nabla_x \Theta(0).e_3) [\nabla_x \Theta(0)]^{-1} \\
&= (u_1|u_2|0) [\nabla_x \Theta(0)]^{-1} - (u_1|u_2|0) (0|0|e_3) [\nabla_x \Theta(0)]^{-1} = (u_1|u_2|0) [\nabla_x \Theta(0)]^{-1}.
\end{aligned} \tag{A.37}$$

vi) Regarding item vi), remark that

$$\begin{aligned}
C_{y_0} &= \text{Cof}(\nabla_x \Theta(0)) \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} [\nabla_x \Theta(0)]^{-1} \\
&= Q_0(0) (\det U_0(0)) U_0^{-1}(0) \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U_0^{-1}(0) Q_0^T(0) = Q_0(0) \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q_0(0)^T,
\end{aligned} \tag{A.38}$$

since  $\det Q_0 = 1$  and

$$(\det U_0(0)) U_0^{-1}(0) \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U_0^{-1}(0) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (\text{A.39})$$

The alternator tensor has the representation Proposition A.3 v), which shows that  $C_{y_0}$  is antisymmetric. Moreover, we deduce

$$\begin{aligned} C_{y_0}^2 &= Q_0(0) \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q_0^T(0) = -Q_0(0) U_0(0) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} U_0^{-1}(0) Q_0^T(0) \\ &= -(\nabla_{y_0}|n_0) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} [\nabla_x \Theta(0)]^{-1} = -(\nabla_{y_0}|0) [\nabla_x \Theta(0)]^{-1} = -A_{y_0}. \end{aligned} \quad (\text{A.40})$$

vii) Let us compute

$$\begin{aligned} Q_0^T(0) \partial_{x_\alpha} Q_0(0) &= \left( d_1^0(0) | d_2^0(0) | d_3^0(0) \right)^T \left( \partial_{x_\alpha} d_1^0(0) | \partial_{x_\alpha} d_2^0(0) | \partial_{x_\alpha} d_3^0(0) \right) \\ &= \begin{pmatrix} 0 & \langle d_1^0(0), \partial_{x_\alpha} d_2^0(0) \rangle & \langle d_1^0(0), \partial_{x_\alpha} d_3^0(0) \rangle \\ \langle d_2^0(0), \partial_{x_\alpha} d_1^0(0) \rangle & 0 & \langle d_2^0(0), \partial_{x_\alpha} d_3^0(0) \rangle \\ \langle d_3^0(0), \partial_{x_\alpha} d_1^0(0) \rangle & \langle d_3^0(0), \partial_{x_\alpha} d_2^0(0) \rangle & 0 \end{pmatrix}, \\ \text{axl}(Q_0^T(0) \partial_{x_\alpha} Q_0(0)) &= \left( -\langle d_2^0(0), \partial_{x_\alpha} d_3^0(0) \rangle | \langle d_1^0(0), \partial_{x_\alpha} d_3^0(0) \rangle | -\langle d_1^0(0), \partial_{x_\alpha} d_2^0(0) \rangle \right)^T, \quad \alpha = 1, 2. \end{aligned} \quad (\text{A.41})$$

Hence, we obtain

$$\begin{aligned} C_{y_0} Q_0(0) &\left( \text{axl}(Q_0^T(0) \partial_{x_1} Q_0(0)) | \text{axl}(Q_0^T(0) \partial_{x_2} Q_0(0)) | 0 \right) [\nabla_x \Theta(0)]^{-1} \\ &= Q_0(0) \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\langle d_2^0(0), \partial_{x_1} d_3^0(0) \rangle & -\langle d_2^0(0), \partial_{x_2} d_3^0(0) \rangle & 0 \\ \langle d_1^0(0), \partial_{x_1} d_3^0(0) \rangle & \langle d_1^0(0), \partial_{x_2} d_3^0(0) \rangle & 0 \\ -\langle d_1^0(0), \partial_{x_1} d_2^0(0) \rangle & -\langle d_1^0(0), \partial_{x_2} d_2^0(0) \rangle & 0 \end{pmatrix} [\nabla_x \Theta(0)]^{-1} \\ &= Q_0(0) \begin{pmatrix} \langle d_1^0(0), \partial_{x_1} d_3^0(0) \rangle & \langle d_1^0(0), \partial_{x_2} d_3^0(0) \rangle & 0 \\ \langle d_2^0(0), \partial_{x_1} d_3^0(0) \rangle & \langle d_2^0(0), \partial_{x_2} d_3^0(0) \rangle & 0 \\ 0 & 0 & 0 \end{pmatrix} [\nabla_x \Theta(0)]^{-1} \\ &= Q_0(0) Q_0^T(0) \left( \partial_{x_1} d_3^0(0) | \partial_{x_2} d_3^0(0) | 0 \right) [\nabla_x \Theta(0)]^{-1} = (\nabla_{n_0}|0) [\nabla_x \Theta(0)]^{-1} = -B_{y_0}. \quad \blacksquare \end{aligned} \quad (\text{A.42})$$

## A.5 Neumann condition on the transverse boundary: an alternative approach

One can also assume that on the transverse boundary (upper and lower face of the fictitious Cartesian configuration  $\Omega_h$ ) the Neumann condition

$$S_1 \left( \nabla_x \varphi(x_1, x_2, \pm \frac{h}{2}), \bar{R}_s(x_1, x_2, \pm \frac{h}{2}) \right) \cdot (\pm e_3) = 0 \quad (\text{A.43})$$

holds. Using (3.31), this implies

$$\langle T_{\text{Biot}} \left( \bar{U}_{e,s}(x_1, x_2, \pm \frac{h}{2}) \right) \cdot n_0, n_0 \rangle = 0. \quad (\text{A.44})$$

Hence, for these boundary conditions, we obtain the following linear algebraic system of equations for the two unknown functions  $\varrho_m^e$  and  $\varrho_b^e$

$$\begin{aligned} \varrho_m^e &\left[ (\lambda + 2\mu) \pm \lambda \frac{h}{2} \langle \bar{Q}_e^T (\nabla(\bar{Q}_e \nabla_x \Theta(0)) \cdot e_3) | 0 \rangle, \left[ \nabla_x \Theta(\pm \frac{h}{2}) \right]^{-T} \right] \\ &+ \varrho_b^e \left[ \pm \frac{h}{2} (\lambda + 2\mu) + \lambda \frac{h^2}{8} \langle \bar{Q}_e^T (\nabla(\bar{Q}_e \nabla_x \Theta(0)) \cdot e_3) | 0 \rangle, \left[ \nabla_x \Theta(\pm \frac{h}{2}) \right]^{-T} \right] \\ &= \lambda + 2\mu - \lambda \left[ \langle \bar{Q}_e^T (\nabla m) | 0 \rangle, \left[ \nabla_x \Theta(\pm \frac{h}{2}) \right]^{-T} \right] - 2]. \end{aligned} \quad (\text{A.45})$$

Further, the dependence of  $\nabla_x \Theta(x_1, x_2, x_3)$  on  $x_3$  will be denoted by  $\nabla_x \Theta(x_3)$ . The above linear algebraic system is equivalent

with the system

$$\begin{aligned}
& \varrho_m^e \left[ 2(\lambda + 2\mu) + \lambda \frac{h}{2} \langle \overline{Q}_e^T (\nabla(\overline{Q}_e \nabla_x \Theta(0).e_3)|0), [\nabla_x \Theta(\frac{h}{2})]^{-T} - [\nabla_x \Theta(-\frac{h}{2})]^{-T} \rangle \right] \\
& + \varrho_b^e \left[ \lambda \frac{h^2}{8} \langle \overline{Q}_e^T (\nabla(\overline{Q}_e \nabla_x \Theta(0).e_3)|0), [\nabla_x \Theta(\frac{h}{2})]^{-1} + [\nabla_x \Theta(-\frac{h}{2})]^{-T} \rangle \right] \\
& = 2(\lambda + 2\mu) - \lambda \langle \overline{Q}_e^T (\nabla m|0), [\nabla_x \Theta(\frac{h}{2})]^{-T} + [\nabla_x \Theta(-\frac{h}{2})]^{-T} - 4 \rangle, \\
& \varrho_m^e \lambda \frac{h}{2} \langle \overline{Q}_e^T (\nabla(\overline{Q}_e \nabla_x \Theta(0).e_3)|0), [\nabla_x \Theta(\frac{h}{2})]^{-T} + [\nabla_x \Theta(-\frac{h}{2})]^{-T} \rangle \\
& + \varrho_b^e \left[ h(\lambda + 2\mu) + \lambda \frac{h^2}{8} \langle \overline{Q}_e^T (\nabla(\overline{Q}_e \nabla_x \Theta(0).e_3)|0), [\nabla_x \Theta(\frac{h}{2})]^{-T} - [\nabla_x \Theta(-\frac{h}{2})]^{-T} \rangle \right] \\
& = -\lambda \langle \overline{Q}_e^T (\nabla m|0), [\nabla_x \Theta(\frac{h}{2})]^{-T} - [\nabla_x \Theta(-\frac{h}{2})]^{-T} \rangle.
\end{aligned} \tag{A.46}$$

The determinant of this system is  $\delta_1(h) = h \delta_2(h)$ , where

$$\begin{aligned}
\delta_2(h) &= 2(\lambda + 2\mu)^2 + \frac{3h}{4} \lambda(\lambda + 2\mu) \langle \overline{Q}_e^T (\nabla(\overline{Q}_e \nabla_x \Theta(0).e_3)|0), [\nabla_x \Theta(\frac{h}{2})]^{-T} - [\nabla_x \Theta(-\frac{h}{2})]^{-T} \rangle \\
& - \frac{h^2 \lambda^2}{4} \langle \overline{Q}_e^T (\nabla(\overline{Q}_e \nabla_x \Theta(0).e_3)|0), [\nabla_x \Theta(\frac{h}{2})]^{-T} \rangle \langle \overline{Q}_e^T (\nabla(\overline{Q}_e \nabla_x \Theta(0).e_3)|0), [\nabla_x \Theta(-\frac{h}{2})]^{-T} \rangle > 0 \text{ for } h \ll 1.
\end{aligned} \tag{A.47}$$

We define the quantities

$$\begin{aligned}
M^{h,+} &= \overline{Q}_e^T (\nabla(\overline{Q}_e \nabla_x \Theta(0).e_3)|0) \left\{ [\nabla_x \Theta(\frac{h}{2})]^{-1} + [\nabla_x \Theta(-\frac{h}{2})]^{-1} \right\}, \\
M^{h,-} &= \overline{Q}_e^T (\nabla(\overline{Q}_e \nabla_x \Theta(0).e_3)|0) \left\{ [\nabla_x \Theta(\frac{h}{2})]^{-1} - [\nabla_x \Theta(-\frac{h}{2})]^{-1} \right\}, \\
N^{h,+} &= \overline{Q}_e^T (\nabla m|0) \left\{ [\nabla_x \Theta(\frac{h}{2})]^{-1} + [\nabla_x \Theta(-\frac{h}{2})]^{-1} \right\}, \\
N^{h,-} &= \overline{Q}_e^T (\nabla m|0) \left\{ [\nabla_x \Theta(\frac{h}{2})]^{-1} - [\nabla_x \Theta(-\frac{h}{2})]^{-1} \right\}.
\end{aligned} \tag{A.48}$$

Then, the exact solution is given by

$$\begin{pmatrix} \varrho_m^e \\ \varrho_b^e \end{pmatrix} = \delta_1(h)^{-1} \begin{pmatrix} (\lambda + 2\mu) + \lambda \frac{h}{8} \langle M^{h,-}, \mathbf{1}_3 \rangle & -\lambda \frac{h}{8} \langle M^{h,+}, \mathbf{1}_3 \rangle \\ -\frac{\lambda}{2} \langle M^{h,+}, \mathbf{1}_3 \rangle & (\lambda + 2\mu) \frac{2}{h} + \frac{\lambda}{2} \langle M^{h,-}, \mathbf{1}_3 \rangle \end{pmatrix} \begin{pmatrix} 2(\lambda + 2\mu) - \lambda \langle N^{h,+}, \mathbf{1}_3 \rangle - 4 \\ -\lambda \langle N^{h,-}, \mathbf{1}_3 \rangle \end{pmatrix}.$$

Consider the quantities  $C_1 = (\nabla y_0|n_0)$ ,  $C_2 = (\nabla n_0|0)$ . Then  $\nabla_x \Theta(x_3) = C_1 + C_2 x_3$ . Moreover

$$[\nabla_x \Theta(x_3)]^{-1} = [\mathbf{1}_3 + x_3 C_1^{-1} C_2]^{-1} C_1^{-1} = [\mathbf{1}_3 - x_3 C_1^{-1} C_2 + x_3^2 (C_1^{-1} C_2)^2 - x_3^3 (C_1^{-1} C_2)^3 + \dots] C_1^{-1}. \tag{A.49}$$

Due to the fact that  $C_1$  and  $C_2$  do not depend on  $x_3$ , we have

$$\begin{aligned}
[\nabla_x \Theta(\frac{h}{2})]^{-1} - [\nabla_x \Theta(-\frac{h}{2})]^{-1} &= -2 \left[ \frac{h}{2} C_1^{-1} C_2 + \left(\frac{h}{2}\right)^3 (C_1^{-1} C_2)^3 + \dots \right] C_1^{-1}, \\
[\nabla_x \Theta(\frac{h}{2})]^{-1} + [\nabla_x \Theta(-\frac{h}{2})]^{-1} &= 2 \left[ \mathbf{1}_3 + \left(\frac{h}{2}\right)^2 (C_1^{-1} C_2)^2 + \left(\frac{h}{2}\right)^4 (C_1^{-1} C_2)^4 + \dots \right] C_1^{-1}.
\end{aligned} \tag{A.50}$$

Hence, we deduce

$$\begin{aligned}
M^{h,+} &= 2 \overline{Q}_e^T (\nabla(\overline{Q}_e \nabla_x \Theta(0).e_3)|0) [\nabla_x \Theta(0)]^{-1} + o(h^2), \\
N^{h,+} &= 2 \overline{Q}_e^T (\nabla m|0) [\nabla_x \Theta(0)]^{-1} + o(h^2), \\
M^{h,-} &= o(h), \quad N^{h,-} = -h \overline{Q}_e^T (\nabla m|0) [\nabla_x \Theta(0)]^{-1} (\nabla n_0|0) [\nabla_x \Theta(0)]^{-1} + o(h^2), \\
\delta_1 &= 2(\lambda + 2\mu)^2 + o(h).
\end{aligned} \tag{A.51}$$

and

$$\begin{aligned}
\varrho_m^e &= \frac{1}{\delta_1(h)} \left\{ [(\lambda + 2\mu) + \lambda \frac{h}{8} \langle M^{h,-}, \mathbf{1}_3 \rangle] [2(\lambda + 2\mu) - \lambda \langle N^{h,+}, \mathbf{1}_3 \rangle - 4] + [\lambda \frac{h}{8} \langle M^{h,+}, \mathbf{1}_3 \rangle] [\lambda \langle N^{h,-}, \mathbf{1}_3 \rangle] \right\}, \\
\varrho_b^e &= \frac{1}{\delta_1(h)} \left\{ [-\frac{\lambda}{2} \langle M^{h,+}, \mathbf{1}_3 \rangle] [2(\lambda + 2\mu) - \lambda \langle N^{h,+}, \mathbf{1}_3 \rangle - 4] - [(\lambda + 2\mu) \frac{2}{h} + \frac{\lambda}{2} \langle M^{h,-}, \mathbf{1}_3 \rangle] [\lambda \langle N^{h,-}, \mathbf{1}_3 \rangle] \right\}.
\end{aligned} \tag{A.52}$$

If we take  $h \rightarrow 0$  (as appropriate for a very thin shell) in the expressions (A.52) of  $\varrho_m^e$  and  $\varrho_b^e$ , then we obtain

$$\begin{aligned}
\varrho_m^e &= \frac{1}{2(\lambda+2\mu)^2} \left\{ (\lambda+2\mu)[2(\lambda+2\mu) - \lambda[2\langle \bar{Q}_e^T(\nabla m|0)[\nabla_x \Theta(0)]^{-1}, \mathbf{1}_3 \rangle - 4]] \right\}, \\
&= 1 - \frac{\lambda}{\lambda+2\mu} [\langle \bar{Q}_e^T(\nabla m|0)[\nabla_x \Theta(0)]^{-1}, \mathbf{1}_3 \rangle - 2], \\
\varrho_b^e &= \frac{1}{2(\lambda+2\mu)^2} \left\{ -\frac{\lambda}{2} \langle 2\bar{Q}_e^T(\nabla(\bar{Q}_e \nabla_x \Theta(0).e_3)|0)[\nabla_x \Theta(0)]^{-1}, \mathbf{1}_3 \rangle \right. \\
&\quad \times [2(\lambda+2\mu) - \lambda[\langle 2\bar{Q}_e^T(\nabla m|0)[\nabla_x \Theta(0)]^{-1}, \mathbf{1}_3 \rangle - 4]] \\
&\quad \left. + 2(\lambda+2\mu)\lambda \langle \bar{Q}_e^T(\nabla m|0)[\nabla_x \Theta(0)]^{-1}(\nabla n_0|0)[\nabla_x \Theta(0)]^{-1}, \mathbf{1}_3 \rangle \right\}, \\
&= -\frac{\lambda}{\lambda+2\mu} \langle \bar{Q}_e^T(\nabla(\bar{Q}_e \nabla_x \Theta(0).e_3)|0)[\nabla_x \Theta(0)]^{-1}, \mathbf{1}_3 \rangle \tag{A.53} \\
&\quad + \frac{\lambda}{\lambda+2\mu} \langle \bar{Q}_e^T(\nabla m|0)[\nabla_x \Theta(0)]^{-1}(\nabla n_0|0)[\nabla_x \Theta(0)]^{-1}, \mathbf{1}_3 \rangle \\
&\quad + \frac{\lambda^2}{(\lambda+2\mu)^2} \langle \bar{Q}_e^T(\nabla(\bar{Q}_e \nabla_x \Theta(0).e_3)|0)[\nabla_x \Theta(0)]^{-1}, \mathbf{1}_3 \rangle [\langle \bar{Q}_e^T(\nabla m|0)[\nabla_x \Theta(0)]^{-1}, \mathbf{1}_3 \rangle - 2].
\end{aligned}$$

Ignoring the term  $\frac{\lambda^2}{(\lambda+2\mu)^2} \langle \bar{Q}_e^T(\nabla(\bar{Q}_e \nabla_x \Theta(0).e_3)|0)[\nabla_x \Theta(0)]^{-1}, \mathbf{1}_3 \rangle [\langle \bar{Q}_e^T(\nabla m|0)[\nabla_x \Theta(0)]^{-1}, \mathbf{1}_3 \rangle - 2]$  we obtain the same expressions for  $\varrho_m^e$  and  $\varrho_b^e$  as in (4.26), but here considering the Neumann condition (A.43) instead of the approximated boundary conditions (3.40). Therefore, our result in (4.26) is correct.