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**Title:**

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**Nr. 01-2013**

**ISSN 1841 – 914X**

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# Stochastic super fast diffusion equations with multiplicative noise

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**Abstract**

In this paper we prove an existence and uniqueness result for the stochastic porous media equation with very singular diffusion and multiplicative noise, by using monotonicity techniques. The multiplicative Gaussian noise is essential in the proof of existence.

**Key word**: stochastic PDE's, monotone operators, super-fast diffusion
AMS [2000] 76S05, 60H15, 82D10

## 1 Introduction

This work is concerned with stochastic non-linear equations corresponding to diffusion process of the type

$$dX(t) = \text{div } (X^{\alpha-1}\nabla X)dt \tag{1}$$

where $\alpha \in (-1,0)$, and $X(t,\xi)$ is the density for the time space coordinates $(t,\xi)$.

This type of diffusion has been observed during experiments using Wisconsin toroidal octupole plasma containment device (see [14]). The same model describes the expansion of a thermalized electron cloud (see [13]).

The phenomena is usually called very singular diffusion and is relevant for the case of small densities. In those circumstances we may consider a restriction to non-negative data since the physical applications deal in general with the situation $X \geq 0$. The physical understanding of fast and super-fast diffusions is assured by the knowledge of some remarkable properties which were studied in [17].

The Cauchy problem for the equation (1) was studied in the deterministic case in [15], by using an $L^1$ approach. For asymptotic behaviour and stability of the solution see [11] and [12].

The interest for the corresponding stochastic differential equation follows from the fact that most natural phenomena exhibit variability which cannot be modelled by using a deterministic approaches. More precisely, natural systems can be represented as stochastic models, and the deterministic description is the underlying motivating example.

The purpose of this paper is to study such equations in the framework of stochastic evolution equations by using monotonicity methods. We can easily see that

$$\text{div }\left(X^{\alpha-1}\nabla X\right)=\frac{1}{\alpha}\ \Delta X^{\alpha}$$

and keeping in mind that $\alpha$ is fixed and negative, we can reformulate corresponding stochastic differential equation as

$$\begin{cases} dX_t+\Delta\left(X_t\right)^{\alpha}dt=X_t dW_t, & \textit{in } (0,T)\times\mathcal{O}, \\ \left(X_t\right)^{\alpha}=0, & \textit{on } (0,T)\times\partial\mathcal{O}, \\ X_0=x, & \textit{in } \mathcal{O}. \end{cases} \tag{2}$$

Here $\mathcal{O}$ is a bounded open subset of $\mathbb{R}^d$ with smooth boundary $\partial\mathcal{O}$ and $W(t)$ is a cylindrical Wiener process on a stochastic basis $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$ taking values in a Hilbert space $H$, defined by $W(t)=\sum\limits_{j=1}^{\infty}\mu_j e_j\beta_j(t)$, where $\mu_j\in\mathbb{R}$, $\left\{\beta_j\right\}_j$ are mutually independent Brownian motions over the same stochastic basis and $\left\{e_j\right\}_j$ are the eigenfunctions of the Laplace operator $\Delta$ with Dirichlet homogeneous boundary conditions on $\partial\mathcal{O}$. The system is normalized in the space $L^2(\mathcal{O})$ and the corresponding eigenvalues are denoted by $\lambda_j$. We also assume that $\sum\limits_{j=1}^{\infty}\mu_j^2\lambda_j^2<\infty$.

We shall denote by $H_0^1(\mathcal{O})$, $H^{-1}(\mathcal{O})$ the standard Sobolev spaces on $\mathcal{O}$ and by $\langle .,.\rangle_1$, $\langle .,.\rangle_{-1}$, $|.|_1$ and $|.|_{-1}$ the corresponding inner products and norms. For $p,\ q\in[1,+\infty]$ by $L_W^q\left((0,T);L^p(\Omega;\ H)\right)$ ($H$ a Hilbert space) we shall denote the space of all $q-$ integrable processes $u:[0,T]\to L^p(\Omega;\ H)$ which are adapted to the filtration $\left\{\mathcal{F}_t\right\}_{t\geq 0}$. By $C_W\left([0,T];L^2(\Omega;\ H)\right)$ we denote the space of all $H-$ adapted processes which are mean square continuous.

We shall denote by $C$ a positive constant independent of the approximations, that may change in the chains of estimates.

Recently, the theory of non-linear stochastic equations was intensively studied for the drift of the form $-\Delta\Psi(X)$ where $\Psi:\mathbb{R}\to\mathbb{R}$ is a maximal monotone operator.

In the case $\Psi(x)=x^{\alpha}$ and $\alpha>1$, the corresponding equation describes the dynamics of fluids in porous media (low diffusions) and their existence, uniqueness and positivity of the solution have already been studied in [6], [7], [9], [18], [19] for the stochastic case. For the deterministic case see [1] and [20].

The case $\Psi(x)=x^{\alpha}$ and $\alpha\in[0,1)$ is relevant in the mathematical modelling of dynamics of an ideal gas in a porous media and, in particular, in plasma fast diffusion model for $\alpha=0$ (see [12]). The existence and uniqueness of a strong solution was studied in [6], [7], [19] for more general non-linear stochastic equations. Finite time extinction is studied in 3 dimensions for $\alpha\in\left[\frac{1}{5},1\right)$ in [8].

The case $\Psi(x) = \log x$ was recently studied in [3]. Note that it corresponds in the first formulation to the situations $\alpha = 0$, since div $\left(X^{-1}\nabla X\right) = \ \Delta(\ln X)$.

For the case $\Psi(x) = x^{\alpha}$ and $\alpha \leq -1$, it has been proved that, even in the deterministic case, there is no solution with finite mass. For a detailed analyse of the deterministic case see [20].

The present paper is concerned with the remaining case $\alpha \in (-1, 0)$ and we shall prove existence and uniqueness of the solution for equation (2) in the following sense.

**Definition 1** *Let $x \in H^{-1}(\mathcal{O})$. An $H^{-1}(\mathcal{O})-$valued continuous $\mathcal{F}_t-$ adapted process $X$ is called solution to the super-fast diffusion equation (2) on $[0,T]$ if*

$$X \in L^2\left(0,T;\ L^2\left(\Omega;\ L^2(\mathcal{O})\right)\right),\ X^{\alpha} \in L^2\left(0,T;\ L^2\left(\Omega;\ H_0^1(\mathcal{O})\right)\right)$$

*and*

$$\begin{aligned}(X(t), e_j)_2 &= (x, e_j)_2 + \int_0^t \int_{\mathcal{O}} \langle \nabla (X^{\alpha}(s)), \nabla e_j \rangle \, d\xi ds \\ &\quad + \sum_{k=1}^{\infty} \int_0^t (\mu_k e_k X(s), e_j)_2 \, d\beta_k(s),\end{aligned}$$

*for all eigenfunctions $e_j$ of the Laplace operator and $\forall\ t \in [0,T]$, $\mathbb{P}-a.s.$ .*

The same type of definition was used for the porous media case in [6].

# 2 The Main Result

We can now formulate the main result of this paper.

**Theorem 2** *For each $x \in L^2(\mathcal{O})$ non-negative a.e. on $\mathcal{O}$, there is a unique solution to the super-fast diffusion equation (2) such that*

$$X \in L^{\infty}\left(0,T;\ L^2\left(\Omega;\ L^2(\mathcal{O})\right)\right) \cap L^2\left(\Omega;\ C\left([0,T];\ H^{-1}(\mathcal{O})\right)\right)$$

*and*

$$X^{\alpha} \in L^2\left(0,T;\ L^2\left(\Omega;\ H_0^1(\mathcal{O})\right)\right).$$

**Proof.** We can easily check that the operator

$$\Psi : \mathbb{R}_+ \to \mathbb{R}_-, \quad \Psi(x) = -x^{\alpha}$$

is monotonically increasing and the range $R(Id + \Psi) = \mathbb{R}$, which assures that it is a maximal monotone operator in $\mathbb{R} \times \mathbb{R}$. We denoted by $Id$ the identity function.

Consequently we can take the Yosida approximation for $\Psi$ in $\mathbb{R}$, which is for each $\lambda > 0$, $\Psi_{\lambda} : \mathbb{R} \to \mathbb{R}_-$

$$\Psi_{\lambda}(x) = \frac{1}{\lambda}\left(x - (Id + \lambda\Psi)^{-1} x\right) = \Psi\left((Id + \lambda\Psi)^{-1} x\right), \quad x \in \mathbb{R}$$

and the corresponding resolvent $J_\lambda : \mathbb{R} \to \mathbb{R}_+ = D(\Psi)$

$$J_\lambda(x) = (Id + \lambda\Psi)^{-1} x, \quad x \in \mathbb{R}.$$

We define the operator $\widetilde{\Psi}_\lambda : \mathbb{R} \to \mathbb{R}$ by

$$\widetilde{\Psi}_\lambda(x) = \Psi_\lambda(x) + \lambda x$$

which is Lipschitz and increasing in $\mathbb{R}$, and also strongly monotone, i.e.

$$\left(\widetilde{\Psi}_\lambda(x) - \widetilde{\Psi}_\lambda(y)\right)(x-y) \geq \lambda (x-y)^2, \quad \forall x, y \in \mathbb{R}$$

and that implies that the inverse $\left(\widetilde{\Psi}_\lambda\right)^{-1}$ is also Lipschitz in $\mathbb{R}$.

We shall also need the potential of $\widetilde{\Psi}_\lambda$, i.e. the l.s.c. proper convex function

$$\widetilde{j}_\lambda(x) = j_\lambda(x) + \lambda \frac{x^2}{2}, \quad \forall x \in \mathbb{R}_+$$

where

$$j(x) = \frac{-1}{\alpha+1} x^{\alpha+1}, \quad \forall x \in \mathbb{R}_+$$

is the potential of $\Psi$ and $j_\lambda$ is the Moreau regularization of $j$ (see Theorem 2.9 from [2] and [16]). It is well known that $j_\lambda$ is differentiable and $j_\lambda' = \Psi_\lambda$ for each fixed $\lambda > 0$.

We can easily check that $\widetilde{j}_\lambda$ verifies for each $\lambda$ the conditions from Proposition 2.10 of [2] and consequently we get that $-\Delta\widetilde{\Psi}_\lambda$ is a maximal monotone operator in $H^{-1}(\mathcal{O}) \times H^{-1}(\mathcal{O})$ with the potential $\varphi_\lambda : H^{-1}(\mathcal{O}) \to \mathbb{R}$ defined by

$$\varphi_\lambda(u) = \begin{cases} \int_{\mathcal{O}} \widetilde{j}_\lambda(u(x))\,dx, & u \in L^1(\mathcal{O}) \text{ and } \widetilde{j}_\lambda(u) \in L^1(\mathcal{O}) \\ +\infty, & \text{otherwise.} \end{cases} \tag{3}$$

Note that $\partial\varphi_\lambda = -\Delta\widetilde{\Psi}_\lambda$.

We shall consider now the approximating equation

$$\begin{cases} dX_\lambda(t) - \Delta\widetilde{\Psi}_\lambda(X_\lambda(t))\,dt = X_\lambda(t)\,dW_t, & \textit{in } (0,T)\times\mathcal{O}, \\ \widetilde{\Psi}_\lambda(X_\lambda) = 0, & \textit{on } (0,T)\times\partial\mathcal{O}, \\ X_\lambda(0) = x, & \textit{in } \mathcal{O}. \end{cases} \tag{4}$$

Since $\widetilde{\Psi}_\lambda$ is monotonically increasing and Lipschitz in $\mathbb{R}$ for each $\lambda$, we have by Theorem 2.12 in [5] that equation (4) has for each $x \in L^2(\mathcal{O})$ an unique strong non-negative solution $X_\lambda \in L^2\left(0,T;\ L^2\left(\Omega;\ L^2(\mathcal{O})\right)\right)$ such that $\widetilde{\Psi}_\lambda(X_\lambda) \in L^2\left(0,T;\ L^2\left(\Omega;\ H_0^1(\mathcal{O})\right)\right)$. Note that the strong solution is also a solution in the sense of Definition 1. We need also to mention that the condition $\widetilde{\Psi}_\lambda(0) = 0$ can be avoided via a translation which is possible since the domain $D(\widetilde{\Psi}_\lambda) = \mathbb{R}$ and the range $R\left(\widetilde{\Psi}_\lambda\right) = \mathbb{R}$.

We intend to pass to the limit for $\lambda \to 0$ in

$$\begin{aligned}(X_\lambda(t), e_j)_2 &= (x, e_j)_2 + \int_0^t \int_{\mathcal{O}} \left\langle \nabla \widetilde{\Psi}_\lambda(X_\lambda(s)), \nabla e_j \right\rangle d\xi ds \\ &\quad + \sum_{k=1}^{\infty} \int_0^t (\mu_k e_k X_\lambda(s), e_j)_2 \, d\beta_k(s),\end{aligned} \tag{5}$$

for all $j \in \mathbb{N}$, $t \in [0,T]$ and $\mathbb{P}-a.s.$. To this purpose we need some a priori estimates.

We shall start by the estimate in the norm $|\ .\ |_{L^2(\mathcal{O})}$. Note that we can not do it directly since the approximating solution does not belong to $H_0^1(\mathcal{O})$ and the formal calculation gives

$$\begin{aligned}&\int_0^t \int_{\mathcal{O}} \left\langle \nabla \widetilde{\Psi}_\lambda(X_\lambda(s)), \nabla X_\lambda(s) \right\rangle d\xi ds \\ =\ & \int_0^t \int_{\mathcal{O}} \left( \langle \nabla \Psi_\lambda(X_\lambda(s)), \nabla X_\lambda(s) \rangle + \lambda |\nabla X_\lambda(s)|^2 \right) d\xi ds \\ =\ & \int_0^t \int_{\mathcal{O}} \left( \Psi'(J_\lambda(X_\lambda(s))) J'_\lambda(X_\lambda(s)) |\nabla X_\lambda(s)|^2 + \lambda |\nabla X_\lambda(s)|^2 \right) d\xi ds \\ =\ & \int_0^t \int_{\mathcal{O}} \left( \frac{-\alpha}{J_\lambda^{1-\alpha}(X_\lambda(s)) - \lambda\alpha} + \lambda \right) |\nabla X_\lambda(s)|^2 \, d\xi ds \geq 0,\end{aligned}$$

(since $\alpha \in (-1,0)$ and $J_\lambda : \mathbb{R} \to D(\Psi) = \mathbb{R}_+$).

In order to do this estimate properly we take a second approximation

$$\begin{cases} dX_\lambda^\varepsilon(t) + A_\lambda^\varepsilon(X_\lambda^\varepsilon(t))\, dt = X_\lambda^\varepsilon(t)\, dW_t, & \text{for } t \geq 0, \\ X_\lambda^\varepsilon(0) = x, \end{cases} \tag{6}$$

where

$$\begin{aligned} A_\lambda(x) &= -\Delta(\Psi_\lambda(x) + \lambda x) = -\Delta \widetilde{\Psi}_\lambda(x) \\ D(A_\lambda) &= \left\{ x \in H^{-1}(\mathcal{O}) \cap L^1(\mathcal{O});\quad \Psi_\lambda(x) + \lambda x \in H_0^1(\mathcal{O}) \right\}, \end{aligned}$$

and $A_\lambda^\varepsilon$ is the Yosida approximation of $A_\lambda$, i.e.

$$A_\lambda^\varepsilon(x) = \frac{1}{\varepsilon}\left( Id - (Id + \varepsilon A_\lambda)^{-1} \right)(x), \quad x \in H^{-1}(\mathcal{O}).$$

We know by classical theory that equation (6) has a unique strong solution and by Lemma 3.4 from [6] we have that, for each $\lambda$

$$\begin{aligned} X_\lambda^\varepsilon &\longrightarrow X_\lambda, \quad \text{strongly in } L^\infty\left(0,T;\ L^2\left(\Omega;\ H^{-1}(\mathcal{O})\right)\right), \\ X_\lambda^\varepsilon &\longrightarrow X_\lambda, \quad \text{in the weak}^\star \text{ topology of } L^\infty\left(0,T;\ L^2\left(\Omega;\ L^2(\mathcal{O})\right)\right), \end{aligned} \tag{7}$$

as $\varepsilon \to 0$.

We can now apply the Itô formula to equation (6) for the function

$$\varphi_\nu(x) = \frac{1}{2}\left|(Id - \nu\Delta)^{-1} x\right|_2^2, \quad \nu \geq 0,$$

and, after taking the expectation and letting $\nu \to 0$, we obtain

$$\mathbb{E}\left|X_\lambda^\varepsilon(t)\right|_2^2 + \mathbb{E}\int_0^t \int_{\mathcal{O}} A_\lambda^\varepsilon\left(X_\lambda^\varepsilon(s)\right) X_\lambda^\varepsilon(s)\, d\xi ds = |x|_2^2 + C\mathbb{E}\int_0^t \int_{\mathcal{O}} \left|X_\lambda^\varepsilon(s)\right|^2 d\xi ds, \tag{8}$$

(for all the details concerning the approximation in $\nu$ see Lemma 3.5 from [6]).

Let us denote by $Y_\lambda^\varepsilon$ the solution to the equation

$$Y_\lambda^\varepsilon - \varepsilon\Delta\widetilde{\Psi}_\lambda\left(Y_\lambda^\varepsilon\right) = X_\lambda^\varepsilon, \quad \widetilde{\Psi}_\lambda\left(Y_\lambda^\varepsilon\right) \in H_0^1(\mathcal{O}), \tag{9}$$

(in other words $Y_\lambda^\varepsilon = (Id + \varepsilon A_\lambda)^{-1}(X_\lambda^\varepsilon)$). Since $\widetilde{\Psi}_\lambda$ is strongly monotone we have that $Y_\lambda^\varepsilon$ is also in $H_0^1(\mathcal{O})$. We can now take the inner product in $L^2(\mathcal{O})$ between (9) and $Y_\lambda^\varepsilon$ and we get

$$\left|Y_\lambda^\varepsilon\right|_2^2 + \varepsilon\int_{\mathcal{O}} \left\langle \nabla\widetilde{\Psi}_\lambda\left(Y_\lambda^\varepsilon\right), \nabla Y_\lambda^\varepsilon \right\rangle d\xi = \int_{\mathcal{O}} X_\lambda^\varepsilon Y_\lambda^\varepsilon d\xi.$$

Since

$$\begin{aligned} &\int_0^t \int_{\mathcal{O}} \left\langle \nabla\widetilde{\Psi}_\lambda\left(Y_\lambda^\varepsilon\right), \nabla Y_\lambda^\varepsilon \right\rangle d\xi ds \\ = &\int_0^t \int_{\mathcal{O}} \left(\frac{-\alpha}{J_\lambda^{1-\alpha}\left(X_\lambda(s)\right) - \lambda\alpha} + \lambda\right) \left|\nabla Y_\lambda^\varepsilon(s)\right|^2 d\xi ds \geq 0, \end{aligned}$$

we have that $\left|Y_\lambda^\varepsilon\right|_2^2 \leq \left|X_\lambda^\varepsilon\right|_2^2$ and then, the second term of the left-hand side of (8) is also positive. Indeed,

$$\begin{aligned} \mathbb{E}\int_0^t \int_{\mathcal{O}} A_\lambda^\varepsilon\left(X_\lambda^\varepsilon(s)\right) X_\lambda^\varepsilon(s)\, d\xi ds &= \frac{1}{\varepsilon}\mathbb{E}\int_0^t \int_{\mathcal{O}} \left(X_\lambda^\varepsilon(s) - Y_\lambda^\varepsilon(s)\right) X_\lambda^\varepsilon(s)\, d\xi ds \\ &\geq \frac{1}{2\varepsilon}\mathbb{E}\int_0^t \int_{\mathcal{O}} \left(\left|X_\lambda^\varepsilon(s)\right|^2 - \left|Y_\lambda^\varepsilon(s)\right|^2\right) d\xi ds \geq 0. \end{aligned}$$

Then, by using Gronwall's inequality in (8) we get that

$$\mathbb{E}\left|X_\lambda^\varepsilon(t)\right|_2^2 \leq C\,|x|_2^2,$$

for $C$ a constant independent of $\varepsilon$ and $\lambda$, and by (7) we obtain that

$$\operatorname*{\mathit{ess}\,sup}_{t\in[0,T]} \mathbb{E}\left|X_\lambda(t)\right|_2^2 \leq C\,|x|_2^2. \tag{10}$$

As a direct consequence, we see that, for each $(t,\omega) \in [0,T] \times \Omega$, we have

$$\int_{\mathcal{O}} |X_\lambda(\xi)|^{\alpha+1} d\xi \leq \left(\int_{\mathcal{O}} |X_\lambda(\xi)|^2 d\xi\right)^{\frac{\alpha+1}{2}} |\mathcal{O}|^{\frac{1-\alpha}{2}}$$

where $|\mathcal{O}|$ is the Lebesgue measure of the bounded set $\mathcal{O}$. Then, by the Young inequality ($ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$ with $\frac{1}{p}+\frac{1}{q}=1$) for $p=\frac{2}{\alpha+1}$ and $q=\frac{2}{1-\alpha}$ we obtain that

$$\int_{\mathcal{O}} |X_\lambda(\xi)|^{\alpha+1} d\xi \leq \frac{\alpha+1}{2}\int_{\mathcal{O}} |X_\lambda(\xi)|^2 d\xi + \frac{1-\alpha}{2}|\mathcal{O}|, \tag{11}$$

and then

$$\underset{t\in[0,T]}{ess\sup}\ \mathbb{E}\int_{\mathcal{O}} |X_\lambda(\xi)|^{\alpha+1} d\xi \leq C\left(|x|_2^2+1\right). \tag{12}$$

For the second estimate we take the Itô formula to equation (4) with

$$\varphi_\lambda(x) = \int_{\mathcal{O}} \widetilde{j}_\lambda(x(\xi))\,d\xi, \quad \forall x \in L^2(\mathcal{O}),$$

where $\widetilde{j}_\lambda$ is the potential of $\widetilde{\Psi}_\lambda$. Note that for $x \in L^2(\mathcal{O})$ non-negative a.e. on $\mathcal{O}$ we have that $x^{\alpha+1} \in L^1(\mathcal{O})$ (and also $\widetilde{j}_\lambda(x)$) and the previous integral is finite for each $\lambda$. It is clear that $\widetilde{j}'_\lambda = \widetilde{\Psi}_\lambda$.

For a detailed justification for the use of the Itô formula in this context see Theorem 2.1 from [3] , Lemma 6 from [10] and keep also in mind that $\langle u,v\rangle_{-1} = \left\langle(-\Delta)^{-1}u, v\right\rangle_2$ and then

$$|A_\lambda(X_\lambda)|_{-1}^2 = \left\langle(-\Delta)^{-1}\left(-\Delta\widetilde{\Psi}_\lambda(X_\lambda)\right), \left(-\Delta\widetilde{\Psi}_\lambda(X_\lambda)\right)\right\rangle_2 = \left|\nabla\widetilde{\Psi}_\lambda(X_\lambda)\right|_2^2.$$

For a different context see the second part of Proposition 3.2 from [4].

We obtain, after taking the expectation, that

$$\begin{aligned} &\mathbb{E}\int_{\mathcal{O}} \widetilde{j}_\lambda(X_\lambda(t))\,d\xi + \mathbb{E}\int_0^t\int_{\mathcal{O}} \left\langle\nabla\widetilde{\Psi}_\lambda(X_\lambda(s)), \nabla\widetilde{j}'_\lambda(X_\lambda(s))\right\rangle d\xi ds \\ = \;& \mathbb{E}\int_{\mathcal{O}} \widetilde{j}_\lambda(x)\,d\xi + \frac{1}{2}\sum_{k=1}^{\infty}\mu_k^2\mathbb{E}\int_0^t\int_{\mathcal{O}} |X_\lambda(s)\ e_k|^2\,\widetilde{j}''_\lambda(X_\lambda(s))\,d\xi ds. \end{aligned}$$

We recall that

$$\begin{aligned} \widetilde{j}'_\lambda(x) &= \widetilde{\Psi}_\lambda(x) \\ \widetilde{j}''_\lambda(x) &= \widetilde{\Psi}'_\lambda(x) = \frac{-\alpha}{J_\lambda^{1-\alpha}(x) - \lambda\alpha} + \lambda \end{aligned}$$

and $|e_k|_{L^\infty(\mathcal{O})} \leq c\lambda_k$ with $\sum_{k=1}^{\infty} \mu_k^2 \lambda_k^2 < \infty$. Now we have that

$$\begin{aligned} & \mathbb{E} \int_{\mathcal{O}} \widetilde{j}_\lambda \left(X_\lambda \left(t\right)\right) d\xi + \mathbb{E} \int_0^t \int_{\mathcal{O}} \left|\nabla \widetilde{\Psi}_\lambda \left(X_\lambda \left(s\right)\right)\right|^2 d\xi ds \\ \leq \; & \mathbb{E} \int_{\mathcal{O}} \widetilde{j}_\lambda \left(x\right) d\xi + C\mathbb{E} \int_0^t \int_{\mathcal{O}} X_\lambda^2 \left(s\right) \left( \frac{-\alpha}{J_\lambda^{1-\alpha} \left(X_\lambda\right) - \lambda\alpha} + \lambda \right) d\xi ds. \end{aligned} \tag{13}$$

For the first term of the left-hand side we have by Theorem 2.9 from [2] that

$$\begin{aligned} \mathbb{E} \int_{\mathcal{O}} \widetilde{j}_\lambda \left(X_\lambda\right) d\xi \; & \geq \; \mathbb{E} \int_{\mathcal{O}} \left( j \left(J_\lambda \left(X_\lambda\right)\right) + \lambda \frac{X_\lambda^2}{2} \right) d\xi \\ & \geq \; -\frac{1}{\alpha+1} \mathbb{E} \int_{\mathcal{O}} \left(J_\lambda \left(X_\lambda\right)\right)^{\alpha+1} d\xi. \end{aligned} \tag{14}$$

The first term of the right hand side is

$$\mathbb{E} \int_{\mathcal{O}} \widetilde{j}_\lambda \left(x\right) d\xi \leq \mathbb{E} \int_{\mathcal{O}} \left( -\frac{1}{\alpha+1} x^{\alpha+1} + \lambda \frac{x^2}{2} \right) d\xi \leq |x|_2^2 . \tag{15}$$

Going back to (13) and replacing (14) and (15) we obtain that

$$\begin{aligned} & \mathbb{E} \int_0^t \int_{\mathcal{O}} \left|\nabla \widetilde{\Psi}_\lambda \left(X_\lambda\right)\right|^2 d\xi ds \leq \frac{1}{\alpha+1} \mathbb{E} \int_{\mathcal{O}} \left(J_\lambda \left(X_\lambda\right)\right)^{\alpha+1} d\xi \\ & + |x|_2^2 + C\mathbb{E} \int_0^t \int_{\mathcal{O}} X_\lambda^{1+\alpha} \frac{\left(-\alpha\right) X_\lambda^{1-\alpha}}{\left(J_\lambda \left(X_\lambda\right)\right)^{1-\alpha} - \lambda\alpha} d\xi ds + \lambda \mathbb{E} \int_0^t \int_{\mathcal{O}} |X_\lambda|^2 d\xi ds. \end{aligned}$$

Since $J_\lambda$ is the resolvent of $\Psi_\lambda$ we have

$$J_\lambda \left(X_\lambda\right) - \lambda \left(J_\lambda \left(X_\lambda\right)\right)^\alpha = X_\lambda \tag{16}$$

and since $J_\lambda \left(X_\lambda\right)$ and $X_\lambda$ are non-negative a.e. on $(0, T) \times \Omega \times \mathcal{O}$ we get that

$$0 \leq X_\lambda \leq J_\lambda \left(X_\lambda\right), \quad \text{a.e. on } \; (0, T) \times \Omega \times \mathcal{O}. \tag{17}$$

On the other hand, if we multiply (16) by $J_\lambda \left(X_\lambda\right)$ and integrate over $\mathcal{O}$ we obtain

$$|J_\lambda \left(X_\lambda\right)|_2^2 - \lambda \int_{\mathcal{O}} \left(J_\lambda \left(X_\lambda\right)\right)^{\alpha+1} d\xi = \int_{\mathcal{O}} X_\lambda \; J_\lambda \left(X_\lambda\right) d\xi.$$

By using another Young inequality ($ab \leq a^2 + \frac{1}{4} b^2$) in the last term of the above equality we get that

$$\frac{3}{4} |J_\lambda \left(X_\lambda\right)|_2^2 \leq |X_\lambda|_2^2 + \int_{\mathcal{O}} \left(J_\lambda \left(X_\lambda\right)\right)^{\alpha+1} d\xi \quad (\text{for } \lambda < 1).$$

Then, arguing as in (11) we get

$$\frac{3}{4}\left|J_\lambda\left(X_\lambda\right)\right|_2^2 \leq \left|X_\lambda\right|_2^2 + \frac{\alpha+1}{2}\left|J_\lambda\left(X_\lambda\right)\right|_2^2 + \frac{2}{1-\alpha}\left|\mathcal{O}\right|$$

and then

$$\left|J_\lambda\left(X_\lambda\right)\right|_2^2 \leq \frac{4}{1-2\alpha}\left(\left|X_\lambda\right|_2^2 + C\right).$$

Consequently

$$\begin{aligned}\int_{\mathcal{O}}\left(J_\lambda\left(X_\lambda\right)\right)^{\alpha+1} d\xi &\leq \frac{\alpha+1}{2}\int_{\mathcal{O}}\left(J_\lambda\left(X_\lambda\right)\right)^2 d\xi + C \\ &\leq C\left(\left|X_\lambda\right|_2^2 + 1\right).\end{aligned} \tag{18}$$

On the other hand, by (17) we have that

$$0 \leq \frac{(-\alpha) X_\lambda^{1-\alpha}}{\left(J_\lambda\left(X_\lambda\right)\right)^{1-\alpha} - \lambda\alpha} \leq (-\alpha)\frac{\left(J_\lambda\left(X_\lambda\right)\right)^{1-\alpha}}{\left(J_\lambda\left(X_\lambda\right)\right)^{1-\alpha} + \lambda(-\alpha)} \leq -\alpha.$$

Now by using (18) and then (10) and (12) we get that

$$\mathbb{E}\int_0^t \int_{\mathcal{O}}\left|\nabla\widetilde{\Psi}_\lambda\left(X_\lambda\right)\right|^2 d\xi ds \leq C\left(\underset{t\in[0,T]}{ess\sup}\mathbb{E}\left|X_\lambda(t)\right|_2^2 + 1\right) \leq C. \tag{19}$$

By (10) and (19) we have

$$\begin{aligned} X_\lambda &\to X, \text{ weak}^* \text{ in } L^\infty\left(0,T;\ L^2\left(\Omega;\ L^2(\mathcal{O})\right)\right) \\ &\qquad \text{weakly in } L^2\left(0,T;\ L^2\left(\Omega;\ L^2(\mathcal{O})\right)\right) \end{aligned}$$

$$\lambda X_\lambda \to 0, \text{ strongly in } L^2\left(0,T;\ L^2\left(\Omega;\ L^2(\mathcal{O})\right)\right)$$

$$\widetilde{\Psi}_\lambda\left(X_\lambda\right) \to \eta, \text{ weakly in } L^2\left(0,T;\ L^2\left(\Omega;\ H_0^1(\mathcal{O})\right)\right).$$

Now, if we pass to the limit for $\lambda \to 0$ in (5), we get that

$$\left(X(t), e_j\right)_2 = \left(x, e_j\right)_2 + \int_0^t \int_{\mathcal{O}}\left\langle\nabla\eta, \nabla e_j\right\rangle d\xi ds + \sum_{k=1}^\infty \int_0^t \left(\mu_k e_k X(s), e_j\right)_2 d\beta_k(s), \tag{20}$$

for all $j \in \mathbb{N}$, $t \in [0,T]$ and $\mathbb{P}-a.s.$.

To conclude the proof of existence it is sufficient to show that

$$\eta(t,\xi,\omega) = \Psi\left(X(t,\xi,\omega)\right),\ a.e. \text{ on } (0,T)\times\Omega\times\mathcal{O}.$$

Since the realization of the operator $\Psi$ is maximal monotone in $L^2((0,T)\times\Omega\times\mathcal{O})$ (eventually via a translation), it is sufficient to check that

$$\limsup_{\lambda\to 0}\mathbb{E}\int_0^t\int_{\mathcal{O}}\widetilde{\Psi}_\lambda(X_\lambda)\ X_\lambda d\xi ds\le\mathbb{E}\int_0^t\int_{\mathcal{O}}\eta\ Xd\xi ds. \tag{21}$$

In order to prove this, we need first to check the strong convergence of $\{X_\lambda\}_\lambda$ in $H^{-1}(\mathcal{O})$. Indeed, by applying the Itô formula in $H^{-1}(\mathcal{O})$ to $X_\lambda-X_\mu$ we obtain via Burkholder-Davis-Gundy inequality for $p=1$ that

$$\mathbb{E}\sup_{t\in[0,T]}|X_\lambda(t)-X_\mu(t)|_{-1}^2\le C\max(\lambda,\mu),$$

for details see [9]. That leads to

$$X_\lambda\to X\ \text{ strongly }\ \text{in } L^2\left(\Omega;C\left([0,T];H^{-1}(\mathcal{O})\right)\right).$$

Then we apply The Itô formula to (4) with the $H^{-1}(\mathcal{O})$ norm and we obtain that

$$\begin{aligned}\limsup_{\lambda\to 0}\mathbb{E}\int_0^T\int_{\mathcal{O}}\widetilde{\Psi}_\lambda(X_\lambda(s))\ X_\lambda(s)\,d\xi ds &\le -\frac{1}{2}\mathbb{E}|X(T)|_{-1}^2+\frac{1}{2}\mathbb{E}|x|_{-1}^2\\ &+\frac{1}{2}\sum_{k=i}^{\infty}\mu_k\mathbb{E}\int_0^t|X(s)\ e_k|_{-1}^2\,ds.\end{aligned}$$

On the other hand if we apply the Itô formula to (20) we get that

$$\begin{aligned}\mathbb{E}\int_0^T{}_{H_0^1(\mathcal{O})}\langle\eta(s),X(s)\rangle_{H^{-1}(\mathcal{O})}\,ds &\le -\frac{1}{2}\mathbb{E}|X(T)|_{-1}^2+\frac{1}{2}\mathbb{E}|x|_{-1}^2\\ &+\frac{1}{2}\sum_{k=i}^{\infty}\mu_k\mathbb{E}\int_0^t|X(s)\ e_k|_{-1}^2\,ds.\end{aligned}$$

Now (21) follows directly and then $X$ is a solution in the sense of Definition 1.

To prove uniqueness of the solution we assume by absurd that there are at least two solutions $X_1$ and $X_2$, and we apply the Itô formula in $H^{-1}(\mathcal{O})$ for $X_1-X_2$. We obtain

$$\begin{aligned}&\frac{1}{2}\mathbb{E}|X_1(t)-X_2(t)|_{-1}^2+\mathbb{E}\int_0^t\int_{\mathcal{O}}(\Psi(X_1)-\Psi(X_2))(X_1-X_2)\,d\xi ds\\ &=\frac{1}{2}\sum_{k=i}^{\infty}\mu_k\mathbb{E}\int_0^t|X(s)\ e_k|_{-1}^2\,ds,\quad t\in[0,T].\end{aligned}$$

Since $\Psi$ is monotonically increasing we obtain via Gronwall's lemma that $X_1=X_2$ a.e. on $(0,T)\times\Omega\times\mathcal{O}$ and the proof is complete. ■

**Acknowledgement 3** *This work was supported by the post-doctoral research project PN-II-RU-PD-2012-3-0240 of the Romanian National Authority for Scientific Research.*

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